GENERAL HELICOIDS WITH POINTWISE 1 – TYPE GAUSS MAP

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Abstract
It is well known that the Gauss map of a constant mean curvature surface is point wise 1-type. In this paper, we show that if the Gauss map of general helicoids is pointwise 1 – type then, it's mean curvature is a constant.

Keywords and Phrases : hypersurface, mean curvature, Gauss map

Résumé. On montre que si le laplacien de l'application de Gauss d'un hélicoide lui est proportionnel alors sa courbure moyenne est constante.

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1 - Introduction

In the framework of the theory of finite type submanifolds (see [1], [2]), the authors of [3] raise the following problem.

Classify all submanifolds in m – Euclidean space E^m (or in E_1^m) satisfying the following equation $\Delta G = fG$ (1)

Where Δ is the Laplacian of the induced metric, G the Gauss map of the map of the submanifold, and for some function *f* on the submanifold.

Definition 1.1. *The Gauss map of an hypersurface is said to be pointwise 1 – type if the condition (1) is satisfied.*

The authors of [3] have classified ruled surfaces in the Minkowski 3 –space E_1^3 with pointwise

1 – Type Gauss map.
In the paper [4], a characterization of the helicoids as ruled surface in the Euclidean 3 – space with pointwise 1 – type Gauss map is obtained.

On the other hand, some classes of submanifolds in the pseudo-Euclidean space with finite Gauss map are studied in [5] and [6]. Choi and Piccini in [7] made a general study on submanifolds of Euclidian spaces with finite type Gauss map and classified the compact surfaces with 1 – type Gaus map.

In the papers [8] and [9], respectively, the rotation surfaces in the Euclidean space E^3 and rotation surfaces in the Minkowski space E_1^3 with pointwise 1 – type Gauss map have been studied and a characterization theorem of them is obtained. In this paper, we generalize the characterization for the class of general helicoids. Let us recall the following. a kinetic property of a rotation surface is its invariance by a rotation around its axis. Such a property is satisfied by cylinders and by the wide class of the general helicoids which contains the cylinders and the rotation surfaces as limit cases.

It is well-known that the Gauss map of a constant mean curvature surface is pointwise 1-type ([6]).

On main result is :

Theorem 1.2. *If the Gauss map of a general helicoids in the Euclidean 3 – space is pointwise 1 – type, then its mean curvature is a constant.*

In this paper, we will assume that all surfaces are connected and all objects are at least of class C^3. We will use freely the notation of vectors by columns or by lines.

2 - Preliminaries

Here, we recall some fundamental formulas for surfaces to be used later in this work. Assume that M is a surface in the Euclidean 3 – dimensional space E^3 with it's canonical metric denoted by $\langle .,. \rangle$.

For two vectors $V = (X_1, X_2, X_3)$ and

$$W = (Y_1, Y_2, Y_3) \text{ in } E^3.$$

$$\langle V, W \rangle = X_1 Y_1 + X_2 Y_2 + X_3 X_3 , (2)$$

and their dot product is given by,

$$V\times W = (X_2 Y_3 - X_3 Y_2, X_3Y_1 - X_1 Y_3, X_1 Y_2 - X_2 Y_1). (3)$$

The surface M may be given locally by an one-to-one isometric immersion X of a open subset U of $\mathbb{R}^2$ into E^3,

$$\begin{array}{rcl} X : \mathcal{U} \subset \mathbb{R}^2 & \longrightarrow & E^3 \\ (s,v) & \longmapsto & X(s,v) \end{array}.$$

And we can identified X(U) with U. So (s, v) are local coordinates on U (see [12]).On U, the Gauss map G is given by the following formulas

$$G = \frac{X_s \times X_v}{\|X_s \times X_v\|}, (4)$$

where $X_s = \frac{\partial X}{\partial s}$ and $X_v = \frac{\partial X}{\partial v}$. The first fundamental form I and the second fundamental form II of the surface M are given in U by

$$\begin{cases} I = \langle X_s, X_s\rangle\, ds^2 + 2\langle X_s, X_v\rangle\, ds\, dv + \langle X_v, X_v\rangle\, dv^2, \\ II = \langle G, X_{ss}\rangle\, ds^2 + 2\langle G, X_{sv}\rangle\, dsdv + \langle G, X_{vv}\rangle\, dv^2 \end{cases} (5)$$

The mean curvature H of the surface is then obtained by the formulas

$$H = \frac{\langle G, X_{ss}\rangle\langle X_v, X_v\rangle - 2\langle G, X_{sv}\rangle\langle X_s, X_v\rangle + \langle G, X_{vv}\rangle\langle X_s, X_s\rangle}{\langle X_s, X_s\rangle\,\langle X_s, X_s\rangle - \langle X_v, X_s\rangle^2} \quad (6)$$

For the Laplacian of the surface M, in local coordinates (x_1, x_2) on U, we have the following formula

$$\Delta = \frac{1}{[\det(g_{ij})]^{\frac{1}{2}}} \sum_{i,j=1} \frac{\partial}{\partial x_j}\left\{\left[\det(g_{ij})\right]^{\frac{1}{2}} g^{ij} \frac{\partial}{\partial x_i}\right\}, (7)$$

where, (g_{ij}) is the matrix of the first fundamental form I of the surface.

3 - Proof of the theorem

Step 1

In this first step, we establish some formulas.

We will use the formulas (4) and (7) above to compute the Gauss map G and the Laplacian ΔG of a general hilicoïd. We will consider that a general helicoids is given by an one-to-one isometric immersion X defined on the open set U of $\mathbb{R}^2$ by

$$X(s, v) = \begin{pmatrix} x(s)\cos v \\ y(s)\sin v \\ z(s) + hv \end{pmatrix}, \text{ where } x(s) > 0,\ h \in \mathbb{R},\ (8)$$

and where the profile curve s → (x(s), 0, z(s)) is parametrised by the arc length s, that is,

$$x'^2 + z'^2 = 1 \quad (9)$$

(see [5]). From $X_s = = \begin{pmatrix} x'\cos v \\ x'\sin v \\ z' \end{pmatrix}$ and

$X_v = \begin{pmatrix} -x\sin v \\ x\cos v \\ h \end{pmatrix}$, we get the first fundamental form

$$I = ds^2 + 2hz'\,dsdv + (x^2 + h^2)\,dv^2;\ (10)$$

and the vector

$$X_s \times X_v = \begin{pmatrix} hx'\sin v - xz'\cos v \\ -xz'\sin v - hx'\cos v \\ xx' \end{pmatrix}.$$

Then we get from (9)

$$\begin{aligned} \|X_s \times X_v\|^2 &= h^2x'^2 + x^2z'^2 + x^2x'^2, \\ &= h^2x'^2 + x^2(x'^2 + z'^2), \\ &= h^2x'^2 + x^2. \end{aligned}$$

It will be convenient to introduce the function

$$\delta = \left(x^2 + h^2x'^2\right)^{1/2}.\ (11)$$

By using the expression of $X_s \times X_v$ given above, we get the Gauss map in the form

$$G = \begin{pmatrix} A(s)\sin v - B(s)\cos v \\ -B(s)\sin v - A(s)\cos v \\ C(s) \end{pmatrix},\ (12)$$

where,

$$A = \frac{hx'}{\delta},\ B = \frac{xz'}{\delta},= \ C = \frac{xx'}{\delta} \qquad (13)$$

satisfy

$$A^2 + B^2 + C^2 = 1. \qquad (14)$$

Now we use the expression of the Gauss map in (12) and the vectors

$$X_{ss} = (x\cos v, x\sin v, z'') \; ;$$
$$X_{sv} = (-x'\sin v, x'\cos v, 0) \; ;$$
$$X_{vv} = (-x\cos v, -x\sin v, 0) \; ;$$

to obtain that,

$$\langle G, X_{ss}\rangle = -x'' B + z'' C \; ;$$
$$\langle G, X_{ss}\rangle = -x' A \; ;$$
$$\langle G, X_{vv}\rangle = x B.$$

$$II = (-x'' B + z'' C)\, ds^2 - 2x' A ds dv + x B dv^2 \quad (15)$$

$$2H = \frac{U(-x''B+z''C) + 2hx'z'A + xA}{\delta^2} \quad (16)$$

Where,

$$U = U(s) = x^2 + h^2. \quad (17)$$

We will get the derivate of the mean curvature in the following equation:

$$2H' = \left[\frac{U(-x'' B + z'' C)}{\delta^2}\right]' + \left[\frac{2hx' z' A + x B)}{\delta^2}\right]' \quad (18)$$

Finally, it remains to find the Laplacian ΔG of the Gauss map G. Since the matrix (g_{ij}) of the first fundamental form *I* is

$$(g_{ij}) = \begin{pmatrix} 1 & hz' \\ hz' & U \end{pmatrix},$$

Then its inverse is

$$(g^{ij}) = \frac{1}{\delta^2}.\begin{pmatrix} U & -hz' \\ -hz' & 1 \end{pmatrix}.$$

Thus

$$\delta\Delta = \frac{\partial}{\partial s}\left[\delta\left(\left(\frac{U}{\delta^2}\right)\frac{\partial}{\partial s} - \left(\frac{hz'}{\delta^2}\right)\frac{\partial}{\partial v}\right)\right] + \frac{\partial}{\partial v}\left[\delta\left(-\left(\frac{hz'}{\delta^2}\right)\frac{\partial}{\delta^2} + \frac{1}{\delta^2}\frac{\partial}{\partial v}\right)\right]$$

By (7)

So, we get easily that

$$\delta\Delta G = \left(\frac{U}{\delta}\right)' G_s + \left(\frac{U}{\delta}\right) G_{ss} - \left(\frac{hz'}{\delta}\right)' G_v - 2\left(\frac{hz'}{\delta}\right) G_{sv} + \left(\frac{1}{\delta}\right) C_{vv} \quad (19)$$

To obtain δΔG, we will need the following six vectors,

$$G = \begin{pmatrix} A\sin v - B\cos v \\ -B\sin v - A\cos v \\ C \end{pmatrix}$$

$$G_s = \begin{pmatrix} A'\sin v - B'\cos v \\ -B'\sin v - A'\cos v \\ C' \end{pmatrix}$$
$$G_v = \begin{pmatrix} B\sin v + A\cos v \\ -A\sin v - B\cos v \\ 0 \end{pmatrix}$$
$$G_{sv} = \begin{pmatrix} B'\sin v + A'\cos v \\ A'\sin v - B'\cos v \\ 0 \end{pmatrix}$$
$$G_{ss} = \begin{pmatrix} A''\sin v - B''\cos v \\ B''\sin v - A''\cos v \\ C'' \end{pmatrix}$$
$$G_{vv} = \begin{pmatrix} -A\sin v - B\cos v \\ B\sin v - A\cos v \\ C \end{pmatrix}$$

So we can see that

$$\delta\Delta G = G_v = \begin{pmatrix} \alpha(s)\sin v - \beta\cos v \\ -\beta\sin v - \alpha\cos v \\ \gamma(s) \end{pmatrix}, (20)$$

Where α, β and γ are given by

$$\begin{cases} \alpha = \left(\frac{U}{\delta}\right)' A' - \left(\frac{hz'}{\delta}\right)' B - 2\left(\frac{hz'}{\delta}\right) B' + \left(\frac{U}{\delta}\right) A'' - \left(\frac{1}{\delta}\right) A\,; \\ \beta = \left(\frac{U}{\delta}\right)' B' - \left(\frac{hz'}{\delta}\right)' A + 2\left(\frac{hz'}{\delta}\right) A' + \left(\frac{U}{\delta}\right) B'' - \left(\frac{1}{\delta}\right) B\,; \\ \gamma = \left(\frac{U}{\delta}\right)' C' + \left(\frac{U}{\delta}\right) C'' \end{cases} \quad (21)$$

These relations can be rewritten as

$$\begin{cases} \alpha = \left[\left(\frac{U}{\delta}\right) A'\right]' - \left(\frac{hz'}{\delta}\right)' B - 2\left(\frac{hz'}{\delta}\right) B' - \left(\frac{1}{\delta}\right) A\,; \\ \beta = \left[\left(\frac{U}{\delta}\right) B'\right]' + \left(\frac{hz'}{\delta}\right)' A + 2\left(\frac{hz'}{\delta}\right) A' + \left(\frac{1}{\delta}\right) B\,; \\ \gamma = \left[\left(\frac{U}{\delta}\right) C'\right]' \end{cases} \quad (22)$$

Remarque 3.1. From the expression of $\delta\Delta G$ given in (20) and that of the Gauss map G in (12), we get that

$$\delta\Delta G, G) = \alpha A + \beta B + \gamma C. \ (23)$$

Which becomes?

$$\begin{cases} \alpha = (\alpha A + \beta B + \gamma C) A\ , \\ \beta = (\alpha A + \beta B + \gamma C) B\ , \\ \gamma = (\alpha A + \beta B + \gamma C)\ C. \end{cases} \quad (24)$$

Step 3

Here we make some remarks for solving the equations in (24).

Since A = $\frac{hx\prime}{\delta}$, we first assume that x' = 0 and h ≠ 0. In this case, the surface is right helicoids. Indeed we have x = x_o a positive number, z = 1 or z = -1, A= C = 0, and B = ± 1.
The right helicoid is a minimal surface, that is, a surface of constant mean curvature zero.
If h = 0, the surface becomes a rotation surface and the problem is solved in [5]. In the remaining parts we will assume that h is not zero and x' is never zero.

Step 4

In this last step, we may assume that A = A(s) = $\frac{hx\prime}{\delta}$ is a non vanishing function on some interval of the real line (by step 3).

a).First, we are going to prove that the condition (24) is equivalent to the two following equations:

$$\beta A = \alpha B; \gamma A = \alpha C \quad (25)$$

From (24), we have $\alpha = \lambda A$, $\beta = \lambda B$, $\gamma = \lambda C$, where, $\lambda = \alpha A + \beta B + \gamma C$.

Then (25) is easily obtained.

Conversely, assume (25) is true. The equation $\beta A = \alpha B$ implies $\alpha = \lambda A$ and $\beta = \lambda B$, for some function $\lambda = \lambda(s)$.

But, since A is never zero, we have $\lambda = \frac{\alpha}{A}$. Then, the second equation in (25) becomes, $\gamma = \lambda C$.
So we have $\alpha = \lambda A$, $\beta = \lambda B$, $\gamma = \lambda C$. Since $A^2 + B^2 + C^2 = 1$

Then $\lambda = \alpha A + \beta B + \gamma C$. So (24) implies (25). Finally, (24) is equivalent to (25).

b).If $\beta A = \alpha B$, then H is constant
Using (22), $\beta A = \alpha B$ is equivalent to

$$\left[\left(\frac{U}{\delta}\right) B'\right]' A \left(\frac{U}{\delta}\right) A' B + \left(\frac{hz'}{\delta}\right)'(A^2 + B^2) + \left(\frac{2hz'}{\delta}\right)(AA' + BB') = 0 \quad (26)$$

$$T_1 + T_2 = 0,$$

We can put equation (26), in the form

Where T1 = $\left[\frac{U}{\delta}(B'A - A'B)\right]'$, T_2 = $\left[\frac{hz'}{\delta}(A^2 + B^2)\right]'$ using the fact

$(B'A - A'B')' = B''A - A''$.

From the equation (13), we get:

$$A' = h\,\frac{x''\delta - x''\delta'}{\delta^2}$$

And

$$B' = \frac{x''z'\delta + x\,z''\delta - x\,z'\delta'}{\delta^2}$$

Then

$$B'A = A'B = \frac{h}{\delta^3}\left\{x'^2\,z'\delta + x\,x'z''\delta - x\,x'z'\delta' + x\,x'z'\delta'\right\}$$

$$= \frac{h}{\delta^2}\ \{x'^2\, z'\ + (xx')\, z''\delta - (x\, z')\, x''\}$$

$$= \frac{h}{\delta^2}\ [C\, z'' -\ B\, x''] + \frac{hx\prime^2 z\prime}{\delta^2} \text{ by (13)}$$

Bearling in mind that T1 = $\left[\frac{U}{\delta}\ (B'A - A'B)\right]'$, we have

$$T_1 = h\left[\frac{U}{\delta^2}\ (C\, z'' -\ B\, x'') + \frac{x'^2 z' U}{\delta^3}\right]'.$$

So, the relation $T_1 + T_2 = 0$ becomes

$$\left[\frac{U}{\delta^2}\ (C\, z'' -\ B\, x'') + \frac{x'^2 z' U}{\delta^3}\right]' + \left[\frac{x'^2\, z' U}{\delta^3} = \frac{z'}{\delta}\ (A^2 +\ B^2)\right]' = 0.$$

Now it remains to compare this equation, with the equation (18) which can be written as

$$\left[\frac{U}{\delta^2}\ (C\, z'' -\ B\, x'')\right]' = 2H' - \left[\frac{2hx'z'A + x\, B}{\delta^2}\right]'$$

We get easily that $T_1 + T_2 = 0$ becomes

$$2H' - \left[\frac{2hx'z'A + x\, B}{\delta^2}\ \right]' + \left[\frac{x'^2\, y'U}{\delta^3} + \frac{z'}{\delta}\ (A^2 +\ B^2\right]' = 0$$

That is $2H'$ + T = 0, where

$$T = \left[\frac{2hx'z'A + x\, B}{\delta^2}\right]' + \left[\frac{x'^2 z'U}{\delta^3} + \frac{z'}{\delta}(A^2 +\ B^2)\right]'$$

Now let us show that T is zero.

Since A = $\frac{xx\prime}{\delta}$, B = $\frac{xz\prime}{\delta}$, U = $x^2 + h^2$, we have:

$$T = \left[\frac{z'}{\delta^3}\left(-2h^2\, x'^2 -\ x^2 + x'^2\ (x^2 +\ h^2) + h^2\, x'^2 + x^2\, z'^2\right)\right]'$$

Finally, by using the fact that $z'^2 = 1 - x'^2$, we see that

$$T = \left[\frac{z'}{\delta^3}\left(-\ x + x'^2\, x^2 +\ x^2\ (1 - x'^2\right)\right]' = 0$$

Hence, the theorem is proved.

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