CHARACTERIZATION OF HOLOMORPHICALLY FILLABLE CONTACT STRUCTURES ON SOME T^2 – BUNDLES OVER S^1

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Abstract.
In this notes we give An estimation up to isotopy of the number of holomorphically fillable contact structures on T^2 - bundle over S^1, with non-periodic monodromy matrix $A \in SL_2$ (Z) satisfying |tr A| = 2.

Keywords and Phrases : T^2-bundles over the S^1, monodromy, hollomorphically fillable.

Résumé.

Dans cet article on donne une estimation, à isotopie près, du nombre de structures de contact holomorphiquement remplissables sur les fibres en tores T^2 sur le cercle S^1 dont la matrice de monodromie est de trace en valeur absolue égale à 2.

0-Introduction

The main Theorem in this paper is the following.

Theorem 0.0.1 Let M *be* a T^2 - bundle over S^1 with non-periodic monodromy matrix $A \in SL_2$ (Z) satisfying |tr A| = 2 and HF_M the number up to isotopy of holomorphically fillable contact structures on *M*. Then we have $1 \leq HF_M \leq 3$.

Before proving the main Theorem 0.0.1 in section 3, we recall in section 1, general notions of contact structures, most of them are taken in papers cited in reference.

In section 2, we discuss tightness and fillability of a contact structure. As an application we recall the result of Y. Eliashberg ([4]), which assert that there exist a unique holomorphically fillable contact structure on the 3-torus with the particularity that here the explicit filling is given.

So the result of this paper can be seen as an extension of the result of Y. Eliashberg in ([4]) to all T^2 - bundle over S^1 with non-periodic monodromy matrix $A \in SL_2$ (Z) satisfying |tr A|= 2.

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2 - Contact manifolds

We recommend as a general reference on contact geometry Geiges' recent textbook ([9]).

Let M be a differential manifold and $\xi \subset TM$ a hyper plane fields on M. Locally ξ can always be written as the kernel of a non-vanishing 1-form α. One way to see this is to choose an auxiliary Riemannian metric g on M and then to define $\alpha = g(X, .)$, where X is a local non-zero section of the line bundle $\xi^{\perp}$ (the orthogonal complement of ξ in TM). We see that the existence of a globally defined 1-form α with ξ= Ker α is equivalent to the orientability (hence triviality) of $\xi^{\perp}$, i.e. the coorientability of ξ. In this paper, the manifold M will be assumed to be oriented and all the plane fields supposed to be coorientable.

If α satisfies the Frobenius integrability condition $\alpha \wedge d\alpha = 0$, then ξ is an integrable hyper plane field (and vice versa), and its integral submanifolds form a codimension 1 foliation of M. And an integrable hyper plane field is locally of the form dp = 0, where p is a coordinate function on M.

Contact structures are in a certain sense the exact opposite of integrable hyper plane fields.

1.1 Basic notions

Definition 1.1.1 *Let M be a (2n+1)-dimensional manifold. A* ***contact structure*** *on M is a hyper plane distribution ξ in TM given by a global 1-form α such that $\alpha \wedge (d\alpha)^n$ vanishes nowhere. We say that the pair (M, ξ) is a* ***contact manifold*** *and that α is a* ***contact form****. The form α is called* ***positive*** *if $\alpha \wedge (d\alpha)^{n}$ defines the chosen orientation of M. If n is odd, then the orientation defined by $\alpha \wedge (d\alpha)^{n}$ does not depend on the choice of the defining form α, hence one can speak about* ***positive contact structures****.*

The following listed 1-forms are contact forms and the verification is left to the reader.

Example 1.1.2 *on R^{2n+1} with Cartesian coordinates $(x_1, y_1, \ldots, x_n, y_n, z)$, the 1-form $\alpha_1 = dz + \sum^{n}_{i=1} x_i\, dy_i$ is a contact form.*

Example 1.1.3 *on R^{2n+1} with polar coordinates (r_j, θ_j) for the (x_j, y_j)-plane, j=1,....,n, the 1-form* $\alpha_1 = dz + \sum^{n}_{j=1} r_j\, d\theta_j = dz + \sum^{n}_{j=1} (x_j\, dy_j - y_j\, dx_j)$ is a contact form.

Example 1.1.4 *on the sphere S^{2n+1}, say S^3. On has a contact form α by restricting on S^3 the 1-form ω_0 on R^4, with coordinates (x_0, y_0, x_1, y_1) defined as follow:*
$\omega_0 = x_0dy_0 - y_0dx_0 + x_1dy_1 - y_1dx_1$.

Let us discuss now the problem of classification of contact structures. As in any problem of classification, one has to decide first which objects are considered equivalent. A particular case of homotopy is obtained by changing the structure using a path of isomorphisms of the underlying fixed space, in which case one speaks about isotopy. Let us be more formal for the case of contact structures.

Definition 1.1.5 *Two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) are called contactomorphic if there is a diffeomorphism $f: M_1 \to M_2$ with $f_*(\xi_1) = \xi_2$, where $f_* : TM_1 \to TM_2$ denotes the differential of f. If ξ_i = Ker α_i, i = 1, 2, this is equivalent to the existence of a nowhere zero function $\lambda : M_1 \to R$ such that $f^* \alpha_2 = \lambda\alpha_1$.*

Example 1.1.6 *the contact manifolds (R^{2n+1}, ξ_i = Ker α_i, i = 1, 2) from the preceding examples are contactomorphic. An explicit contactomorphism f with $f^* \alpha_2 = \alpha_1$ is given by: $f(x; y; z) = [(x + y) / 2, (y - x) / 2, z + x y / 2]$, where x and y stand for*

$(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, respectively, and $x\,y$ stands for $\sum_j x_j y_j$.

Similarly, both these contact structures are contactomorphic to Ker $(dz - \sum_j x_j y_j)$. Any of these contact structures is called the ***standard contact structure*** *on R^{2n+1}.*

Definition 1.1.7 *A* ***homotopy*** *between two contact structures is a smooth path of contact structures connecting them.*

An ***isotopy*** *between two contact structures is a homotopy of the form $(\phi^*_t\xi)_t$, where $(\phi_t)_t$ is a smooth path of self-diffeomorphisms of M.*

Two contact structures ξ and ξ' on M are ***homotopic****, resp.* ***isotopic****, resp.* ***isomorphic*** *if there is a homotopy, resp. an isotopy, resp. a contactomorphic of M which sends ξ on ξ' .*

One usually tries to classify contact structures on a given manifold up to isotopy or up to contactomorphism. Any contact structure may be seen as a hyper plane field, but one has to be careful because a homotopy between the underlying hyper plane fields of two contact structures is not necessarily a homotopy of contact structures.

Again, a general problem of classification of structures splits into a local and into a global one. Like complex structures and foliations, contact forms have no local invariants:

Theorem 1.1.8 *(Darboux) Any contact form may be written in suitable local coordinates as the standard contact structure on R^{2n+1}.*

Globally the situation is distinct, due to the fact that there is a canonical vector field attached to any contact form.

Definition 1.1.9 *Associated with a contact form α one has the so-called Reeb vector field R_α, defined by the equations:*

(i) $d\alpha(R_\alpha, .) = 0$

(ii) $\alpha(R_\alpha) = 1$.

Then any dynamical invariants of the Reeb vector field are invariants of the contact form, which makes one, feel that by deforming a form, the global structure may change drastically. When one keeps instead of the whole contact form only the contact structure defined by a contact form, the situation is completely different.

Theorem 1.1.10 *(Gray [14]) two homotopic contact structures on a closed manifold are isotopic.*

1.2 Contact structures and Sasakian metrics

Definition 1.2.1 *An almost contact structure on a differentiable manifolds M is a triple (Z, η, Φ), where Φ is a tensor field of type (1, 1) (i.e. an endomorphism of TM), Z is a vector field, and α is a 1-form which satisfy $\eta(Z) = 1$ and $\Phi \circ \Phi = -I + Z \otimes \eta$, where I is the identity endomorphism on TM. A smooth manifold with such a structure is called an* ***almost contact manifold****.*

We have seen that contact geometry exists only in odd dimensions. But it interacts very deeply with an even-dimensional geometry, namely symplectic geometry.

Definition 1.2.2 *A* ***symplectic form*** *on an even-dimensional vector space is a non-degenerate exterior form of degree 2. A* ***symplectic form*** *on an even-dimensional manifold is a closed non-degenerate smooth form of degree 2. A* ***symplectic manifold*** *is a manifold endowed with a symplectic form.*

Let (M, α) be a contact manifold with a contact 1-form α and consider $\xi = \text{Ker}\ \alpha \subset TM$. The subbundle ξ is maximally non-integrable and it is called the contact distribution. As a first example of the presence of symplectic structures in the contact world, note that part of Definition 1.1.1 may be rephrased as saying that α is a contact form if and only if $d\alpha$ is a symplectic form in restriction to Kerα. An equivalent formulation of the contact condition is that the pair (ξ, $d\alpha|_\xi$) gives ξ the structure of a symplectic vector bundle. We denote by $J(\xi)$ the space of all almost complex structures J on ξ that are compatible with $d\alpha|_\xi$, that is the subspace of smooth sections J of the endomorphism bundle End(ξ) that satisfy:

$$J^2 = -I,\ d\alpha(JX, JY) = d\alpha(X, Y),$$
$$d\alpha(X; JX) > 0 \quad (1)$$

for any smooth sections X, Y of ξ.

Notice that each $J \in J(\xi)$ defines a Riemannian metric g_ξ on ξ by setting:

$$g_\xi(X, Y) = d\alpha(X, JY). \quad (2)$$

One easily checks that g_ξ satisfies the compatibility condition $g_\xi(JX, JY) = g_\xi(X, Y)$. Furthermore, the map $J \to g_\xi$ is one-to-one, and the space $J(\xi)$ is contractible. A choice of J gives M an almost CR structure.

Moreover, by extending J to all of TM one obtains an almost contact structure. There are some choices of conventions to make here. We define the section Φ of End(TM) by $\Phi = J$ on ξ and $\Phi R_\alpha = 0$, where R_α is the Reeb vector field associated to α. We can also extend the transverse metric g_ξ to a metric g on all of M by

$$g(X, Y) = g_\xi(X, Y) + \alpha(X)\,\alpha(Y) = d\alpha(X, \Phi Y) + \alpha(X)\,\alpha(Y), \quad (3)$$

for all vector fields X, Y on M. One easily also sees that g satisfies the compatibility condition $g(\Phi X, \Phi Y) = g(X, Y) - \alpha(X)\,\alpha(Y)$.

Definition 1.2.3 *A contact manifold M with a contact form α, a vector field R_α, a section Φ of End(TM), and a Riemannian metric g which satisfy the conditions*

$$\alpha(R_\alpha) = 1 \text{ and } \Phi \circ \Phi = -I + R_\alpha \otimes \alpha,$$
$$g(\Phi X, \Phi Y) = g(X, Y) - \alpha(X)\,\alpha(Y)$$

is known as a ***contact metric*** *on M.*

Definition 1.2.4 *A contact metric manifold (M, α) is called a* ***Sasakian*** *manifold if the Reeb vector field R_α of α is a Killing vector field of unit length on M so that the tensor field Φ of type (1, 1), defined by $\Phi(X) = -\nabla_X R_\alpha$, satisfies the condition*

$$(\nabla_X \Phi)(Y) = g(X, Y)\, R_\alpha - g(R_\alpha, Y)\, X$$

for any pair of vector fields X and Y on M and where ∇ denote the Levi-Civita connection associated to the contact metric.

The quadruple $S = (R_\alpha, \alpha, \Phi, g)$ is called a **Sasakian structure** on *M*.

Definition 1.2.5 *Let (M, S) be a Sasakian manifold and let ϕ: M $\to$ M be a diffeomorphism, then (M, ϕ^*S) is a isomorphic Sasakian structure where*

$$\phi^*S = (\phi^{-1}{}_*R_\alpha, \phi^*\alpha, \phi^{-1}{}_*\Phi\phi_*, \phi^*g).$$

1.3 Contact structures coming from complex geometry.

Another very important class of examples (which is the central one for the results presented here) comes from complex analytic geometry.

Start from a connected complex manifold X of complex dimension $n \geq 2$ and from a real smooth hypersurface M of it.

Denote by J: TX $\to$ TX the (integrable) almost complex structure associated to the complex structure of X, where TX denotes the tangent bundle of the underlying smooth manifold of X.

Then J(TM) cannot be equal to TM, because this last space is odd-dimensional.

Therefore $\xi := TM \cap J(TM)$ is a J-invariant subspace of real codimension 1 of TM, that is, a hyper plane distribution with a natural complex structure $J|_\xi$. We will call it the complex distribution of $M \to X$.

For various hypersurfaces M one can get all the degrees of integrability of this distribution, from the completely integrable case till the completely non-integrable (or contact) one. A general situation when ξ is automatically contact is got when M is strongly pseudoconvex.

Definition 1.3.1 *Let ρ be a smooth function on X. It is called* ***strictly plurisub - harmonic*** *(abbreviated* ***spsh****) if* $- d (d^c\rho) > 0$, *where* $d^c \rho : = d\rho \circ J \in T^*X$.

If a cooriented real hypersurface of X may be defined locally in the neighborhood of any of its points as a regular level of a spsh function which grows from its negative to its positive side, then it is called ***strongly pseudoconvex****.*

It is important to take care about the coorientation of the hypersurface: seen from one side it is pseudoconvex, from the other it is pseudoconcave. The terminology was chosen such that the positive side is the pseudoconcave one, distinguished by the fact that holomorphic curves tangent to the hypersurface are locally contained in that side.

The announced general family of contact manifolds given by complex geometry is presented in the next proposition:

Proposition 1.3.2 *([21]) The complex distribution of any strongly pseudoconvex hypersurface of a complex manifold is a (naturally oriented) contact structure.*

The simplest example of this type of construction is given by $X= C^{n+1}$, with $n \geq 1$ and $\rho : = \sum^{n+1}_{j=1} | z_j |^2$. This is a proper spsh function. The complex distribution on any euclidean sphere centered at the origin is therefore a contact structure. As homotheties centered at the origin leave both the foliation of C^{n+1}-{0} by such spheres and the almost complex structure invariant, they realize contactomorphisms between all such contact spheres. Therefore, one gets a well-defined contact structure on S^{2n+1}, called the standard contact structure on it.

2 - Tightness and fillability

There is a strong relationship between contact topology and symplectic topology due to the fact that contact structures provide natural boundary conditions for symplectic structures on manifolds with boundary. Given a contact manifold (M, ξ) and a symplectic manifold (W, ω) with $\partial W = M$, we say that (W, ω) fills (M, ξ) if some compatibility conditions are satisfied. Depending on how restricting these conditions are, there are several different notions of fillability.

In the following we will always assume that M is an oriented 3-manifold and ξ is oriented and positive. This means that ξ is the kernel of a globally defined smooth 1-form α on M such that $\alpha \wedge d\alpha$ is a volume form inducing the fixed orientation of M.

Definition 2.0.3 *The contact structure ξ on M is called tight if there is no embedded disc $D \subset M$ such that its boundary ∂D is tangent to ξ while D is transversal to ξ along the boundary.*

Tightness of a contact structure is guaranteed by the following properties of fillability.

2.1 Various notions of fillability.

Let us come back to the examples of contact manifolds originating in complex geometry.

Definition 2.1.1 *A complex manifold with a proper spsh function is called a* ***Stein manifold****.*

Consider a Stein manifold X and a proper spsh function $\rho : X \rightarrow R$ bounded from below. Let $M := X_{\rho=a}$ be a regular level of ρ. We call the compact sublevel $Y := X_{\rho\leq a}$ a compact Stein manifold. One should note that Y is a compact smooth manifold with-boundary, but that it is not a compact complex manifold. By Proposition 1.3.2, the complex distribution on M is a contact structure. This motivates:

Definition 2.1.2 *A contact manifold which is contactomorphic to the contact boundary of a compact Stein manifold is called* ***Stein fillable****, and any such compact Stein manifold is a* ***Stein filling*** *of the initial manifold.*

A more general notion is obtained by asking that the bounded from below and proper function ρ be spsh only in a neighborhood of its considered regular level M. One obtains then the notion of compact complex manifold with boundary, and a related notion of filling:

Definition 2.1.3 *A contact manifold which is contactomorphic to the complex distribution on a strongly pseudoconvex boundary of a compact complex manifold with boundary is called* ***holomorphically fillable****, and any such compact complex manifold is a* ***holomorphic filling*** *of the initial manifold.*

Holomorphically fillable contact structure can be found in the world of Sasakian manifold by a result of Marinescu and Yeganefar ([20]), which assert that:

Proposition 2.1.4 *([20]) every compact Sasakian manifold is holomorphically fillable.*

One may forget part of the previous structures in order to arrive at concepts of symplectic geometry, which make no reference to an almost-complex structure:

Definition 2.1.5 *A* ***strong symplectic filling*** *of (M, ξ) is a compact symplectic manifold (Y, ω) with boundary ∂Y = M such that there exists a primitive α of ω in a neighborhood of M whose restriction to M is a defining form of ξ.*

A ***weak symplectic filling*** *of (M, ξ) is a compact symplectic manifold (Y, ω) with boundary ∂Y = M such that the restriction of ω to ξ is a field of positive symplectic forms on ξ.*

Thus, we have four different rigidity notions for contact structures, which are ordered as follows:

Proposition 2.1.6 *holomorphically fillable ⇒ Stein fillable ⇒ strongly symplectically fillable ⇒ weakly symplectically fillable ⇒ tight.*

Proof. Weakly fillable contact structures are tight by a deep theorem of Eliashberg and Gromov ([5, 15]). And by the Definition (2.1.5), we see that strongly symplectically fillable are weakly symplectically fillable. If (M, ξ) is a holomorphically fillable contact 3-manifold, then it is necessarily Stein fillable (Bogomolov and de Oliveira ([3])), so there exists a Stein manifold X and a proper spsh function

$\rho : X \rightarrow R$ bounded from below such that $M: = X_{\rho=a}$ be a regular level of ρ. If one denotes:

$$\alpha_\rho := - d^c\rho$$
$$\omega_\rho := d\alpha_\rho.$$

Then $\alpha_\rho|_M$ is a contact form defining ξ, ω_ρ is a symplectic form on X and $(X_{\rho\leq a}, \omega_\rho)$ is a compact symplectic manifold with boundary $\partial X_{\rho\leq a} = M$, thus (M, ξ) is strongly symplectically fillable by the Definition (2.1.5).

However, it is know now that the four different rigidity notions do not coincide: tight but non weakly fillable contact structures have been found first by Etnyre and Honda ([8]) and later by Lisca and Stipsicz ([18, 19]). A weakly fillable but no strongly fillable contact structure has been found first by Eliashberg ([4]) and later more have been found by Ding and Geiges ([10]). And strongly fillable contact 3-manifolds without Stein fillings have been found by Ghiggini in ([12]).

2.2 Fillable contact structure on T^3

Most of the results in this section can be found in ([4]). The standard contact structure ζ_1 on T^3 is the contact structure on the unit cotangent bundle of the 2-torus. If (x, y) are cyclic coordinates in T^2, and θ is a 1-periodic coordinate along the fiber S^1, then ζ_1 can be defined by the 1-form $\alpha_1 = \cos\theta\, dx + \sin\theta\, dy$.

Proposition 2.2.1 ***the standard contact structure ζ_1 is holomorphically fillable***.

Proof. Indeed, as every cotangent space $T^*T^2 = T^2 \times R^2$ is equipped with a natural symplectic form $\omega_0 = d\lambda$, where λ is the Liouville 1-form $\lambda = z_1dx + z_2dy$, where $(x, y)\in T^2 = R^2/Z^2$ and $(z_1, z_2) \in R^2$. Let $(r,\theta)\in]0, +\infty[\times R/2\pi Z$ be the polar coordinates defined on $R^2 \backslash\{0\}$ by $z_1 = r \cos\theta$ and $z_2 = r \sin\theta$, the restriction of the Liouville 1-form λ on $W_0 = T^2 \times (R^2 \backslash\{0\})$, is the 1-form $\alpha_r = r (\cos\theta\, dx + \sin\theta\, dy)$.

α_r induces in each hypersurface: $r = f(x, y, \theta)$, where f is a positive function, a contact form and all these contact forms, define the same contact structure ζ_1 on $T^3 = \{(x, y, \theta)\}$. The symplectization of (T^3, ζ_1) is isomorphic to $(W_0, \omega_0 = dz_1 \wedge dx + dz_2 \wedge dy)$.

Moreover the Liouville vector field $X_0 = z_1 (\partial/\partial z_1) + z_2 (\partial/\partial z_2)$.

is ω_0 - dual to λ and can be taken gradient-like to a spsh function on W_0. Thus, by the equivalent Definitions (2.1.2) and (2.1.3), (T^3, ζ_1) is holomorphically fillable.

Let p_n: $T^3 \to T^3$, n = 2… be a sequence of cyclic coverings $(x, y, \theta) \to (x, y, n\theta)$. Let us denote by ζ_n, n = 1… the pullback of the contact structure ζ_1 under the covering p_n. Thus, $\zeta_n = \{\alpha_n = 0\}$ where $\alpha_n = p^*_n\alpha_1 = \cos (n\theta)\, dx + \sin (n\theta)\, dy$.

All the contact structures ζ_n are clearly tight because they all have the same universal covering: the standard contact structure on R^3. Moreover, as noticed by E. Giroux in ([13]), all these structures are weakly symplectically fillable. Also E. Giroux ([13]), and independently Y. Kanda ([16]), proved the following theorem.

Theorem 2.2.2 *([13; 16]) the contact structures ζ_n, n = 1… are pairwise non diffeomorphic and give the complete, up to diffeomorphism, list of positive tight contact structures on T^3.*

Furthermore Eliashberg proved in ([4]) the following theorem.

Theorem 2.2.3 *([4]) the contact structures ζ_n for n > 1 are not strongly symplectically fillable, and therefore not holomorphically fillable.*

Combining the Theorems (2.2.3) and (2.2.2) and the Proposition (2.2.1), Eliashberg get in ([4]) the following corollary.

Corollary 2.2.4 *([4]) the contact structure ζ_1 on T3 is the unique up to isotopy holomorphically fillable contact structure.*

3 - Proof of the main Theorem

The set HHF_M of symplectically fillable contact structures considered up to homotopy (as cooriented 2-plane fields) on a closed 3-manifold M is a subtle invariant of M. By a theorem of Eliashberg and Thurston ([7]), the cardinality of HHF_M is an upper bound for the number of homotopy classes of taut foliations on M. And in ([17]) P. Lisca proved the following.

Proposition 3.0.5 ([17]) *let M be a closed oriented 3-manifold carrying metrics with strictly positive scalar curvature. Then,* $|HHF_M| \leq |\text{Tor } H_1(M, Z)|$, *where Tor* $H_1(M, Z)$ *is the torsion subgroup of* $H_1(M, Z)$ *and* $|\ |$ *denotes cardinality.*

The following corollary is our direct consequence of the Propositions (2.1.6) and (3.0.5).

Corollary 3.0.6 *let M be a closed oriented 3-manifold carrying metrics with strictly positive scalar curvature. Then,* $|HF_M| < |\text{Tor } H_1(M, Z)|$, *(4) where* HF_M *be the number, up to isotopy, of holomorphically fillable contact structure on M.*

The 3-dimensional Heisenberg group Nil^3 can be described by the group of 3 by 3 real matrices of the form (see ([2])). As a manifold it is just R^3.

There are two natural isomorphic Sasakian structures on Nil^3: the **right invariant contact form** $\alpha^R = dz - y\,dx$, and the **left invariant contact form** $\alpha^L = dz - x\,dy$. These are related by the involution:

$$\iota : R^3 \to R^3 \text{ defined by } \iota(x, y, z) = (y, x, z), \text{ that is } \iota^*\alpha^L = \alpha^R.$$

Notice that ι reverses orientation. These contact forms give rise to the **right invariant Sasakain structure** $S^R = (Z, \alpha^R, \Phi^R, g^R)$ and the **left invariant Sasakain** structure SL = $(Z, \alpha^L, \Phi^L, g^L)$, where $Z = \partial_z$,

$$\Phi^R = (\partial_x + y\,\partial_z) \otimes dy - \partial_y \otimes dx,\ g^R = (dx)^2 + (dy)^2 + (dz - y\,dx)^2,$$

and

$$\Phi^L = (\partial_y + x\,\partial_z) \otimes dx - \partial_x \otimes dy,\ g^L = (dx)^2 + (dy)^2 + (dz - x\,dy)^2.$$

Both Sasakian structures have the same Reeb vector field. So the Heisenberg group has what we called a **bi-Sasakian structure**. Moreover we see that

$$S^R = (\iota^{-1}{}_* Z, \iota^*\alpha^L, \iota^{-1}{}_*\Phi^L\iota^*, \iota^*g^L) = \iota^*S^L.$$

Thus it follows from the Definition 1.2.5, that S^L and S^R are isomorphic.

Therefore we can fix one of these structures, namely the right Sasakian structure S^L and refer to it as the **standard Sasakian or CR structure** on Nil^3. Moreover it is proved in ([2]) that:

Proposition 3.0.7 *([2]) Let M be a 3-dimensional compact manifold, which is diffeomorphic to a left quotient of the 3-dimensional Heisenberg group* Nil^3, *then the only Sasakian structure passes down to the quotient is the standard one.*

Proof of the main Theorem (0.0.1)

Let M be a T^2 - bundle over S^1 with non-periodic monodromy matrix $A \in SL_2(Z)$, satisfying $|\text{tr } A| = 2$. Following Geiges and Gonjalo ([11]), M is a left quotient of Nil^3. Moreover from ([2]), M is diffeomorphic to the quotient manifold formed by the subgroup $\Gamma_k = Nil^3(Z, k)$ of Nil^3 obtained by restricting the real coordinates (x, y, z) in Nil^3 to take values in the set of all integers divisible by the integer $k > 0$. Furthermore one has always following ([2]) that:

$$H_1(M, Z) = Z \oplus Z_k, \text{ with } k = 1, 2, 3. \text{ Thus } |\text{Tor } H_1(M, Z)| \in \{1, 2, 3\}.$$

Since M is a Sasakian manifold then it follows from ([1]), that it carries a metric with positive scalar curvature. Then, by the above Corollary (3.0.6) we have:

$$0 < HF_M \leq 3.$$

Following the Proposition (3.0.7), the only Sasakian structure passes down to the quotient is the standard one. And by the Proposition (2.1.4) tell us that the induced Sasakian structure is holomorphically fillable, and then by the Proposition 2.1.4, we have:

$$1 \leq HF_M \leq 3.$$

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