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Properties of K-contact manifolds

Propriétés des variétés de K-contact

Philippe Rukimbira *

Abstract

This paper is a survey of some of the author's results on K-contact manifolds. More results from others have also been included as far as they are related to the author's own results.

Key words: K-contact, Sasakian.

Résumé

Cet article est un passage en revue de certains résultats de l'auteur sur les variétés de K-contact. D'autres résultats de différents auteurs ont été ajoutés à mesure qu'ils sont reliés à ceux de l'auteur.

Mots clés : K-contact, Sasakienne.

1 Introduction

The paper is organized as follows. In section 2, we deal with preliminaries on contact metric structures, including some constructions of contact metrics. Section 3 includes some curvature characterization of Sasakian structures. The 1-nullity distribution is introduced and a description provided for its leaf dimension on Sasakian manifolds. The angle function is also defined. This function is somehow involved in the description of the first basic cohomology on K-contact manifolds, leading to the nonexistence results for K-contact structures on odd-dimensional tori and parallel 1-forms on K-contact manifolds. The most general result about nonexistence of parallel forms in K-contact geometry is stated without proof, which uses more than just the angle function.

2 Preliminaries

A contact form on a $2n+1$ dimensional manifold M is a 1-form α such that the identity $\alpha \wedge (d\alpha)^n \neq 0$ holds everywhere on M. Given such a 1-form α, there is always a unique vector field Z satisfying $\alpha(Z) = 1$ and $i_Z d\alpha = 0$. The vector field Z is called the characteristic vector field of the contact manifold (M, α) and the corresponding 1-dimensional foliation is called a contact flow.

2.1 Examples of almost contact structures

We will give examples of contact forms since almost contact structures are always present whenever contact forms are.

1. $\mathbb{R}^{2n+1}$: $\alpha = dz - \sum_{i=1}^{n} y^i dx^i$ and $Z = \frac{\partial}{\partial z}$.
2. The sphere S^3 : on $\mathbb{R}^4$, with coordinates (x^0, y^0, x^1, y^1), consider $\omega_0 = x^0dy^0 - y^0dx^0 + x^1dy^1 - y^1dx^1$. Let η be the restriction of ω_0 to S^3. We claim that $\eta \wedge d\eta \neq 0$ on S^3. Indeed,

$$\eta \wedge d\eta = 2(x^0dy^0 - y^0dx^0 + x^1dy^1 - y^1dx^1) \wedge (dx^0 \wedge dy^0 + dx^1 \wedge dy^1).$$

*Department of Mathematics & Statistics, Florida International University, Miami, FL 33199, USA. rukim@fiu.edu.

Notice that the 1-form $\beta = x^0dx^0 + x^1dx^1 + y^0dy^0 + y^1dy^1$ is normal to S^3 and easily,

$$\beta \wedge \eta \wedge d\eta = ((x^0)^2 + (x^1)^2 + (y^0)^2 + (y^1)^2)(dx^0 \wedge dy^0 \wedge dx^1 \wedge dy^1) \neq 0$$

along S^3 which shows that $\eta \wedge d\eta$ is nowhere zero on S^3.

The above example generalizes to $S^{2n+1} \subset \mathbb{R}^{2n+2}$ with coordinates $(x^0, ..., x^n, y^0, ..., y^n)$ and to convex hypersurfaces in symplectic manifolds $\Sigma \to (M^{2n}, \Omega)$, $L_N\Omega = \Omega$, where N represents the outer unit normal vector field along Σ and Ω is the symplectic form on M.

3. Another example is T^3 with coordinates $\theta^1, \theta^2, \theta^3 : \eta = \cos\theta^3 d\theta^1 + \sin\theta^3 d\theta^2$.

The $2n$ dimensional distribution $D(p) = \{v \in T_pM : \alpha(p)(v) = 0\}$, which is invariant by Z, is called the contact distribution. It carries a $(1,1)$ tensor field J such that $-J^2$ is the identity on D. The tensor field J extends to all of TM if one requires $JZ = 0$. Also, the contact manifold (M, α) carries a nonunique Riemannian metric g adapted to α and J in the sense that the following identities are satisfied for any vector fields X and Y on M.

$$g(X,Y) = g(JX,JY) + \alpha(X)\alpha(Y) \quad (1)$$

$$d\alpha(X,Y) = 2g(X,JY) \quad (2)$$

$$J^2X = -X + \alpha(X)Z; \quad JZ = 0 \quad (3)$$

Such a metric g is called a contact metric.

Our convention for the differential of a 1-form is as follows:

$$d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]).$$

2.2 Construction of contact metric structures

Given (M, α, Z), let g_0 be any metric on M. Let g_1 be the metric equal to g_0 on $D = kern\ \alpha$, $g_1(Z,Z) = 1$ and $g_1(D,Z) = 0$. One defines a skew symmetric tensor field A as follows: $d\alpha(X,Y) = 2g_1(AX,Y)$ for any sections X, Y of D. When restricted to D, A is clearly nonsingular. Let $B = \sqrt{AA^*}$, where A^* is the g_1 adjoint of A. B is a symmetric, positive definite endomorphism of D which commutes with A and A^*. On D, we now define $J = -B^{-1}A$. Clearly $J^2 = B^{-1}AB^{-1}A = B^{-2}AA = A^{*-1}A = -A^{-1}A = -Id$ shows that J is an almost complex structure on D. Observe that J is g_1-orthogonal. For horizontal X and Y,

$$\begin{aligned} g_1(B^{-1}AX, B^{-1}AY) &= g_1(AX, B^{-2}AY) \\ &= -g_1(X, AB^{-2}AY) \\ &= -g_1(X, J^2Y) \\ &= g_1(X,Y) \end{aligned}$$

We extend J by $JZ = 0$ and define an intermediary metric g_2 as follows:

$$g_2(X,Y) = \frac{1}{2}[g_1(JX,JY) + g_1(J^2X, J^2Y)] + \alpha(X)\alpha(Y).$$

Finally, a contact metric g is obtained as: $g(X,Y) = g_2(BX,Y) + \alpha(X)\alpha(Y)$. Using the identity $J^3 = -J$, we will verify identities (1), (2) and (3). First,

$$\begin{aligned} g(JX,JY) &= g_2(BJX,JY) \\ &= \frac{1}{2}[g_1(JBJX, J^2Y) + g_1(J^2BJX, J^2JY)] \\ &= \frac{1}{2}[g_1(J^2BX, J^2Y) + g_1(JBX, JY)] \\ &= g_2(BX,Y) = g(X,Y) - \alpha(X)\alpha(Y). \end{aligned}$$

Next,

$$\begin{aligned} 2g(X,JY) &= 2g_2(BX,JY) \\ &= g_1(JBX,J^2Y)+g_1(J^2BX,J^3Y) \\ &= g_1(AX,Y)+g_1(JAX,JY) \\ &= g_1(AX,Y)+g_1(AX,Y) \\ &= \frac{1}{2}d\alpha(X,Y)+\frac{1}{2}d\alpha(X,Y)=d\alpha(X,Y) \end{aligned}$$

Finally, $J^2X = J^2((X-\alpha(X)Z)+\alpha(X)Z) = -(X-\alpha(X)Z) = -X+\alpha(X)Z$. It is easy to see that $L_Z\alpha = 0$ and $L_Z d\alpha = 0$, but $L_Z J$ and $L_Z g$ need not vanish!

Proposition 2.1. *On a contact metric manifold (M,α,J,g), $L_ZJ=0$ if and only if $L_Zg=0$.*

Proof.

$$\begin{aligned} L_Zg(X,JY) &= Zg(X,JY)+g([Z,X],JY)+g(X,(L_ZJ)Y)+g(X,J[Z,Y]) \\ &= \frac{1}{2}Zd\alpha(X,Y)-g([Z,X],JY)-g(X,(L_ZJ)Y)-g(X,J[Z,Y]) \\ &= \frac{1}{2}d\alpha([Z,X],Y)+\frac{1}{2}d\alpha(X,[Z,Y])-\frac{1}{2}d\alpha([Z,X],Y) \\ &\quad -g(X,(L_ZJ)Y)-\frac{1}{2}d\alpha(X,[Z,Y]) \\ &= -g(X,(L_ZJ)Y) \end{aligned}$$

Therefore, if $L_Zg=0$, then $(L_ZJ)Y=0$ for arbitrary Y. Conversely, suppose $L_ZJ=0$. Then, from the above observation, $L_Zg(X,JY)=0$ for any Y. We need to show that $L_Zg(X,Z)=0$ for all X.

$$\begin{aligned} L_Zg(X,Z) &= Zg(X,Z)-g([Z,X],Z) \\ &= Z\alpha(X)-g([Z,X],Z) \\ &= \alpha([Z,X])-\alpha([Z,X])=0. \end{aligned}$$

This completes the proof. □

A contact metric structure on which $L_ZJ=0$ is called a K-contact structure. From the above proposition, the characteristic vector field of a K-contact metric structure is an infinitesimal isometry, also known as a Killing vector field.

Lemma 2.2. *On a K-contact manifold (M,α,Z,J,g), one has $\nabla_YZ=-JY$, for all vector field Y on M.*

Proof.

$$\begin{aligned} 2g(X,JY) &= d\alpha(X,Y)=X\alpha(Y)-Y\alpha(X)-\alpha([X,Y]) \\ &= Xg(Z,Y)-Yg(Z,X)-g(Z,[X,Y]) \\ &= g(\nabla_XZ,Y)-g(\nabla_YZ,X) \\ &= -2g(X,\nabla_YZ). \end{aligned}$$

So $JY=-\nabla_YZ$. □

We shall adopt the convention $R(X,Y)W=\nabla_X\nabla_YW-\nabla_Y\nabla_XW-\nabla_{[X,Y]}W$, for the Riemann curvature tensor.

Proposition 2.3. *On a K-contact manifold (M,α,Z,J,g), the following identity holds :*

$$(\nabla_XJ)Y=R(Z,X)Y. \tag{4}$$

Proof. Using the above lemma:

$$\begin{aligned}
g(R(Z,X)Y,W) &= g(\nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y, W)\\
&= g(\nabla_Z\nabla_X Y - \nabla_X[Z,Y] - \nabla_X\nabla_Y Z - \nabla_{[Z,X]}Y, W)\\
&= g(\nabla_X JY, W) + g(\nabla_Z\nabla_X Y, W) - g(\nabla_X[Z,Y], W)\\
&\quad -g(\nabla_{[Z,X]}Y, W)\\
&= g(\nabla_X JY, W) + Zg(\nabla_X Y, W) - g(\nabla_X Y, \nabla_Z W)\\
&\quad -g(\nabla_X[Z,Y], W) - g(\nabla_{[Z,X]}Y, W)\\
&= g(\nabla_X JY, W) - g(\nabla_X[Z,Y], W)\\
&\quad +g([Z,\nabla_X Y], W) - g(J\nabla_X Y, W) - g(\nabla_{[Z,X]}Y, W).
\end{aligned}$$

Let ψ_t denote the 1-parameter group of isometries generated by Z. Then

$$\begin{aligned}
[Z,\nabla_X Y] &= -\frac{d}{dt}_{|t=0}\psi_{t*}\nabla_X Y = -\frac{d}{dt}_{|t=0}\nabla_{\psi_{t*}X}\psi_{t*}Y\\
&= \nabla_{[Z,X]}Y + \nabla_X[Z,Y].
\end{aligned}$$

Hence, the above calculation is continued as: $g(R(Z,X)Y,W) = g((\nabla_X J)Y, W)$. Since W was arbitrary, we conclude that $R(Z,X)Y = (\nabla_X J)Y$. □

Given a contact metric structure (M,α,Z,J,g), consider the product manifold $M\times\mathbb{R}$. A vector field on $M\times\mathbb{R}$ can be written as $X + f\frac{d}{dt}$ where X is tangent to M, t is the coordinate on $\mathbb{R}$ and f is a smooth function on $M\times\mathbb{R}$. We define an almost complex structure ϕ on $M\times\mathbb{R}$ by:

$$\phi(X + f\frac{d}{dt}) = JX - fZ + \alpha(X)\frac{d}{dt}.$$

If ϕ is a complex structure, we say that the contact structure (α, Z, J) is normal and the corresponding contact metric structure is called Sasakian.

By a classic theorem of Newlander and Nirenberg, an almost complex structure ϕ of class C^∞ is a complex structure if and only if its Nijenhuis torsion $[\phi,\phi]$ vanishes. The Nijenhuis torsion $[T,T]$ of a tensor field T of type $(1,1)$ is a tensor field given by:

$$[T,T](X,Y) = T^2[X,Y] + [TX,TY] - T[TX,Y] - T[X,TY].$$

It can be directly verified that

$$[T,T](fX,Y) = f[T,T](X,Y) \text{ and } [T,T](X+W,Y) = [T,T](X,Y) + [T,T](W,Y)$$

for any vector fields X, Y, W and smooth function f. It is clear that the Nijenhuis torsion $[\phi,\phi]$ of ϕ vanishes if and only if $[\phi,\phi](X,Y) = 0$ and $[\phi,\phi](X,\frac{d}{dt}) = 0$ for any vector fields X and Y tangent to M. First, we evaluate $[\phi,\phi](X,Y)$:

$$\begin{aligned}
[\phi,\phi](X,Y) &= -[X,Y] + [JX + \alpha(X)\frac{d}{dt}, JY + \alpha(Y)\frac{d}{dt}]\\
&\quad -\phi[JX + \alpha(X)\frac{d}{dt}, Y] - \phi[X, JY + \alpha(Y)\frac{d}{dt}]\\
&= -[X,Y] + [JX,JY] + (JX\alpha(Y) - JY\alpha(X))\frac{d}{dt}\\
&\quad -\phi[JX,Y] + \phi(Y\alpha(X)\frac{d}{dt}) - \phi[X,JY] - \phi(X\alpha(Y)\frac{d}{dt})\\
&= J^2[X,Y] - \alpha([X,Y])Z + [JX,JY] - J[JX,Y]\\
&\quad -J[X,JY] - (\alpha([JX,Y]) + \alpha([X,JY]) - JX\alpha(Y)\\
&\quad +JY\alpha(X)\frac{d}{dt} + (X\alpha(Y) - Y\alpha(X))Z\\
&= [J,J](X,Y) + d\alpha(X,Y)Z + (JX\alpha(Y) - JY\alpha(X)\\
&\quad -\alpha([JX,Y]) - \alpha([X,JY]))\frac{d}{dt}.
\end{aligned}$$

Next we evaluate $[\phi,\phi](X,\frac{d}{dt})$:

$$\begin{aligned}
[\phi,\phi](X,\frac{d}{dt}) &= [JX+\alpha(X)\frac{d}{dt},-Z]-\phi[JX+\alpha(X)\frac{d}{dt},\frac{d}{dt}]-\phi[X,-Z]\\
&= -[JX,Z]+Z\alpha(X)\frac{d}{dt}+J[X,Z]+\alpha([X,Z])\frac{d}{dt}\\
&= (L_ZJ)X+L_Z\alpha(X)\frac{d}{dt}.
\end{aligned}$$

The identity $L_Z\alpha=0$ is valid on any contact structure, therefore, a contact structure is normal if and only if the following 3 identities are satisfied for any X and Y.

$$[J,J](X,Y)+d\alpha(X,Y)Z = 0 \tag{5}$$

$$JX\alpha(Y)-JY\alpha(X)-\alpha([JX,Y])-\alpha([X,JY]) = 0 \tag{6}$$

$$(L_ZJ)X = 0 \tag{7}$$

Proposition 2.4. *Identity (5) implies identities (6) and (7). Therefore, a contact structure (α,Z,J) is normal if and only if $[J,J](X,Y)+d\alpha(X,Y)Z=0$.*

Proof. Setting $Y=Z$ in (5),we obtain:

$$\begin{aligned}
0=[J,J](X,Z) &= J^2[X,Z]-J[JX,Z]\\
&= -[X,Z]+\alpha([X,Z])Z-J[JX,Z]\\
&= -[X,Z]-\alpha([Z,X])Z+J(L_ZJ)X+J^2[Z,X]\\
&= J(L_ZJ)X.
\end{aligned}$$

Hence, applying J on both sides $0=J^2(L_ZJ)X=-(L_ZJ)X+\alpha((L_ZJ)X)Z=-(L_ZJ)X$. This proves that (5) implies (7). To prove the implication (5)⇒(6), apply α to the identity

$$\begin{aligned}
0 &= [J,J](JX,Y)+d\alpha(JX,Y)Z\\
&= -J^2[JX,JY]+[J^2X,JY]-J[J^2X,Y]-J[JX,JY]+d\alpha(JX,Y)Z.
\end{aligned}$$

$$\begin{aligned}
0 &= \alpha([J^2X,Y])+d\alpha(JX,Y)\\
&= \alpha([-X,JY])+\alpha(X)\alpha([Z,JY])-JY\alpha(X)+d\alpha(JX,Y)\\
&= -\alpha([X,JY])-JY\alpha(X)+d\alpha(JX,Y)\\
&= -\alpha([X,JY])-JY\alpha(X)+JX\alpha(Y)-\alpha([JX,Y]).
\end{aligned}$$

This proves the implication (5)⇒ (6). □

It follows from the above proposition that a Sasakian contact metric structure is K-contact. The converse holds in dimension 3; that is,

Proposition 2.5. *A K-contact 3-dimensional manifold is Sasakian.*

Proof. Let Z,E,JE be a local adapted orthonormal frame field on a K-contact 3-dimensional manifold (M,α,Z,J,g). In order to prove that the structure is Sasakian, it is enough to show that $[J,J](E,E)=0$, $[J,J](E,JE)+d\alpha(E,JE)Z=0$ and $[J,J](E,Z)=0$.

$$[J,J](E,E)=-J[JE,E]-J[E,JE]=-J[JE,E]+J[JE,E]=0$$

$$\begin{aligned}
[J,J](E,JE)+d\alpha(E,JE)Z &= J^2[E,JE]+[JE,J^2E]-J[E,J^2E]+d\alpha(E,JE)Z\\
&= -[E,JE]+\alpha([E,JE])Z-[JE,E]-\alpha([E,JE])Z\\
&= 0
\end{aligned}$$

$$\begin{aligned}
[J,J](E,Z) &= J^2[E,Z]-J[JE,Z]=-[E,Z]+\alpha([E,Z])Z-J[JE,Z]\\
&= -[E,Z]+J(L_ZJ)E+J^2[Z,E]\\
&= -[E,Z]-J^2[E,Z]=0.
\end{aligned}$$

□

Recall on a K-contact manifold, $(\nabla_X J)Y = R(Z, X)Y$. More generally, for a contact metric structure (J, Z, α, g), the covariant derivative of J is given by:

$$2g((\nabla_X J)Y, A) = g(N^1(Y, A), JX) + d\alpha(JY, X)\alpha(A) - d\alpha(JA, X)\alpha(Y).$$

Proof. Recall these identities:

$$2g(\nabla_X Y, A) = Xg(Y, A) + Yg(A, X) - Ag(X, Y) + g([X, Y], A) + g([A, X], Y) - g([Y, A], X)$$

and

$$d\Phi(X, Y, A) \quad = \quad X\Phi(Y, A) - Y\Phi(X, A) + A\Phi(X, Y) - \Phi([X, Y], A) - \Phi([Y, A], X) + \Phi([X, A], Y).$$

Therefore,

$$\begin{aligned}
2g((\nabla_X J)Y, A) &= 2g(\nabla_X JY, A) + 2g(\nabla_X Y, JA)\\
&= Xg(JY, A) + JYg(A, X) - g(X.JY) + g([X, JY], A) + g([A, X], JY) - g([JY, A], X)\\
&\quad +Xg(Y, JA)+Yg(JA, X)-JAg(X, Y)+g([X, Y], JA)+g([JA, X], Y)-g([Y, JA], X)\\
&= X\frac{1}{2}d\alpha(A, Y) + JY[\frac{1}{2}d\alpha(JA, X) + \alpha(A)\alpha(X)] - A\frac{1}{2}d\alpha(X, Y)\frac{1}{2}d\alpha(J[X, JY], A)\\
&\quad +\alpha([X, JY])\alpha(A) + \frac{1}{2}d\alpha([A, X], Y) - \frac{1}{2}d\alpha(J[JY, A], X) - \alpha([JY, A])\alpha(X)\\
&\quad +X[\frac{1}{2}d\alpha(Y, A)] + Y\frac{1}{2}d\alpha(X, A) - JA[\frac{1}{2}d\alpha(JX, Y) + \alpha(X)\alpha(Y)] + \frac{1}{2}d\alpha([X, Y], A)\\
&\quad +\frac{1}{2}d\alpha(J[JA, X], Y) + \alpha([JA, X])\alpha(Y) - \frac{1}{2}d\alpha(J[Y, JA], X) - \alpha([Y, JA])\alpha(X)\\
&= \frac{1}{2}d\alpha([A, Y], X) + \frac{1}{2}d\alpha([JY, JA], X) + \alpha([X, JY])\alpha(A) + \frac{1}{2}d\alpha([JY, A], JX)\\
&\quad -\alpha([JY, A])\alpha(X) + JY[\alpha(A)\alpha(X)] - JA[\alpha(X)\alpha(Y)] + \alpha([JA, X])\alpha(Y)\\
&\quad +\frac{1}{2}d\alpha([Y, JA], JX) - \alpha([Y, JA])\alpha(X)\\
&= \frac{1}{2}d\alpha(-[Y, A] - J[JY, A] + [JY, JA] - J[Y, JA], X) + \alpha(A)[\alpha([X, JY]) + JY\alpha(X)]\\
&\quad +\alpha(X)[JY\alpha(A)] - \alpha([JY, A]) + \alpha(Y)[\alpha([JA, X]) - JA(\alpha(X))]\\
&\quad -\alpha(X)[JA(\alpha(Y)) + \alpha([Y, JA])]\\
&= \frac{1}{2}d\alpha(N^{(1)}(Y, A) - d\alpha(Y, A)Z, X) + d\alpha(JY, X)\alpha(A)\\
&\quad +d\alpha(JY, A)\alpha(X) - \alpha(Y)d\alpha(JA, X) - \alpha(X)d\alpha(JA, Y)\\
&= g(N^{(1)}(Y, A), JX) + d\alpha(JY, X)\alpha(A) - d\alpha(JA, X)\alpha(Y)
\end{aligned}$$

where $N^{(1)}(Y, A) = [J, J](Y, A) + d\alpha(Y, A)Z$. □

Theorem 2.6. *On a contact metric manifold (M, α, Z, J, g), the structure is Sasakian if and only if the identity $(\nabla_X J)Y = g(X, Y)Z - \alpha(Y)X$ holds.*

Proof.

$$\begin{aligned}
[J, J](X, Y) &= J^2[X, Y] + [JX, JY] - J[JX, Y] - J[X, JY]\\
&= J^2(\nabla_X Y - \nabla_Y X) + \nabla_{JX} JY - \nabla_{JY} JX - J(\nabla_{JX} Y - \nabla_Y JX + \nabla_X JY - \nabla_{JY} X)\\
&= J(J\nabla_X Y - \nabla_X JY) - J(J\nabla_Y X - \nabla_Y JX) + \nabla_{JX} JY - J\nabla_{JX} Y - \nabla_{JY} JX + J\nabla_{JY} X\\
&= J((\nabla_Y J)X - (\nabla_X J)Y) + (\nabla_{JX} J)Y - (\nabla_{JY} J)X\\
&= (J\nabla_Y J - \nabla_{JY} J)X - (J\nabla_X J - \nabla_{JX} J)Y
\end{aligned}$$

So if $(\nabla_X J)Y = g(X,Y)Z - \alpha(Y)X$, then

$$\begin{aligned}
[J,J](X,Y) &= J(\nabla_Y J)X - (\nabla_{JY} J)X - J(\nabla_X J)Y + (\nabla_{JX} J)Y \\
&= J(-\alpha(X)Y) - (-\alpha(X)JY - JY) - J(-\alpha(Y)X) + (-\alpha(Y)JX) - \alpha(X)JY \\
&\quad +\alpha(X)JY + \alpha(Y)JX - \alpha(Y)JX - g(X,JY)Z + g(Y,JX)Z \\
&= (g(Y,JX) - g(X,JY))Z \\
&= d\alpha(Y,X)Z
\end{aligned}$$

So $[J,J](X,Y) + d\alpha(X,Y)Z = 0$ and (M,α,Z,J,g) is Sasakian.

Conversely, we show that the Sasakian condition implies the identity $(\nabla_X J)Y = g(X,Y)Z - \alpha(Y)X$. Earlier, we proved the following identity:

$$2g((\nabla_X J)Y, A) = g(N^{(1)}(Y,A), JX) + d\alpha(JY,X)\alpha(A) - d\alpha(JA,X)\alpha(Y).$$

So if the structure is Sasakian, then

$$\begin{aligned}
2g((\nabla_X J)Y, A) &= 2g(JY,JX)g(Z,A) - 2g(JA,JX)g(Z,Y) \\
&= 2(g(Y,X) - \alpha(Y)\alpha(X))g(Z,A) - 2(g(A,X) - \alpha(A)\alpha(X))g(Z,Y) \\
&= 2g(X,Y)g(Z,A) - 2g(A,X)g(Z,Y) + 2\alpha(A)\alpha(X)\alpha(Y) - 2\alpha(Y)\alpha(X)\alpha(A) \\
&= 2g(g(X,Y)Z - g(Z,Y)X, A)
\end{aligned}$$

Therefore $(\nabla_X J)Y = g(X,Y)Z - \alpha(Y)X$ as desired. □

2.3 The standard sasakian structure on S^{2n+1}

Let S^{2n+1} be the unit sphere in $\mathbb{C}^{n+1}$ with ν as outer unit normal : $i\colon S^{2n+1} \to \mathbb{C}^{n+1}$, $\nu = Ji_*\xi$ for some tangent vector ξ. Define ϕ and η by $Ji_*X = i_*\phi X + \eta(X)\nu$. Applying J again,

$$-i_*X = i_*\phi^2 X + \eta(\phi(X)\nu - \eta(X)i_*\xi.$$

Hence, $\phi^2 = -I + \eta \otimes \xi$ and $\eta \circ \phi = 0$. From $Ji_*\xi = i_*\phi\xi + \eta(\xi)\nu$, we deduce that $\nu = i_*\phi\xi + \eta(\xi)\nu$ and hence, $\phi\xi = 0$ and $\eta(\xi) = 1$. Therefore, (ϕ,ξ,η) is an almost contact structure.

Denoting by $\tilde{g}$ the standard metric on $\mathbb{C}^{n+1}$ and $g = i^*\tilde{g}$, then

$$g(X,Y) = \tilde{g}(Ji_*X, Ji_*Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y).$$

This shows that (ϕ,ξ,η,g) is an almost contact metric structure on the unit sphere. Denoting by ν the outward unit vector field along the sphere and by $\tilde{\nabla}$ the covariant derivative in Euclidean space, we recall that the second fundamental form σ of the unit sphere is given by: $\sigma(X,Y) = -g(X,Y)\nu$ and $\tilde{\nabla}_X \nu = X$. One has then:

$$\begin{aligned}
0 &= (\tilde{\nabla}_X J)Y \\
&= \tilde{\nabla}_X(\phi Y + \eta(Y)\nu) - J(\nabla_X Y - g(X,Y)\nu) \\
&= \nabla_X \phi Y - g(X,\phi Y)\nu + (X\eta(Y))\nu + \eta(Y)X - \phi\nabla_X Y - \eta(\nabla_X Y)\nu - g(X,Y)\xi \\
&= (\nabla_X \phi)Y - g(X,Y)\xi + \eta(Y)X + ((\nabla_X \eta)(Y) - g(X,\phi Y))\nu
\end{aligned}$$

Taking the tangential part, we see that $(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X$. Hence we will prove that the structure isl Sasakian as soon as we show that η is in fact a contact form.

Setting $Y = \xi$ gives $-\phi\nabla_X\xi = \eta(X)\xi - X$, hence $\nabla_X \xi = -\phi X$. Therefore:

$$\begin{aligned}
d\eta(X,Y) &= X\eta(Y) - Y\eta(X) - \eta([X,Y]) \\
&= Xg(\xi,Y) - Yg(\xi,X) - g(\xi,[X,Y]) \\
&= g(\nabla_X\xi, Y) - g(\nabla_Y\xi, X) \\
&= g(-\phi X, Y) + g(\phi Y, X) \\
&= 2g(X,\phi Y)
\end{aligned}$$

showing that η is a contact form.

The above construction extends to hypersurfaces in Kahler manifolds, as stated in the following result of Tashiro [8].

Theorem 2.7. *Let M^{2n+1} be a hypersurface of a Kahler manifold $\tilde{M}^{2n+2}$. Then the induced almost contact metric structure (ϕ, ξ, η, g) is Sasakian if and only if the second fundamental form σ satisfies: $\sigma = (-g + \beta(\eta \otimes \eta))\nu$ for some function β.*

3 Topology of sasakian manifolds

The above theorem indicate the strong possibility of characterizing Sasakian structures by curvature tensors. The following proposition contains a curvature characterization of Sasakian structures analogous to the K-contact version found in Proposition 2.3.

Proposition 3.1. *A contact metric structure is Sasakian if and only if the following identity holds:*

$$R(X,Y)Z = \alpha(Y)X - \alpha(X)Y$$

Proof. By Theorem 2.6, the Sasakian condition implies

$$\begin{aligned} R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= -\nabla_X JY + \nabla_Y JX + J[X,Y] \\ &= -(\nabla_X J)Y - J\nabla_X Y + (\nabla_Y J)X + J\nabla_Y X + J[X,Y] \\ &= -(\nabla_X J)Y + (\nabla_Y J)X \\ &= -g(X,Y)Z + \alpha(Y)X + g(X,Y)Z - \alpha(X)Y \\ &= \alpha(Y)X - \alpha(X)Y \end{aligned}$$

Next we prove $R(X,Y)Z = \alpha(Y)X - \alpha(X)Y$ implies that the structure is Sasakian. Letting $Y = Z$ in the above identity shows that each sectional curvature including the Reeb field Z is equal to one, a necessary and sufficient condition for K-contactness. (See [1]). Next, the K-contact condition implies $(\nabla_X J)Y = R(Z,X)Y$(See Proposition 2.3). Therefore,

$$\begin{aligned} g((\nabla_X J)Y, A) &= g(R(Z,X)Y, A) \\ &= g(R(Y,A)Z, X) \\ &= g(\alpha(A)Y - \alpha(Y)A, X) \\ &= g(X,Y)g(Z,A) - g(Z,Y)g(X,A) \\ &= g(g(X,Y)Z - \alpha(Y)X, A) \end{aligned}$$

So $(\nabla_X J)Y = g(X,Y)Z - \alpha(Y)X$ □

3.1 The k-nullity distribution

A sub-manifold N in a contact manifold (M, α, Z, J) is said to be invariant if Z is tangent to N and JX is tangent to N whenever X is. An invariant submanifold is of course a contact submanifold. For a real number k, the k-nullity distribution of a Riemannian manifold (M, g) is the subbundle $N(k)$ defined at each point $p \in M$ as follows: $N_p(k) = \{H \in T_pM : R(X,Y)H = k(g(Y,H)X - g(X,H)Y\}$, for any $X, Y \in T_pM$. Proposition 3.1 says that a contact metric structure is Sasakian if and only if its Reeb vector field belongs to the 1-nullity distribution. If $H \neq 0$ is in $N(k)$, then the sectional curvatures of all plane sections containing H are equal to k. The distribution $N(k)$ is known to be integrable with totally geodesic leaves of constant curvature k. Hence, if $k > 0$ and the dimension of $N(k)$ is > 1, then each leaf of $N(k)$ is a compact submanifold. We refer to [4] for the proof of the following result about the dimension of the 1-nullity distribution's leaves.

Theorem 3.2. *On a closed Sasakian $2n+1$-dimensional manifold, the dimension of $N(1)$ is either less or equal to n, or it is equal to $2n+1$.*

3.2 The angle function on K-contact manifolds

Suppose U is a horizontal Killing vector field on a Sasakian manifold. Then JU is a Killing vector field which is a section of the 1-nullity distribution, hence U itself belongs to the 1-nullity distribution since the later is totally geodesic and $-JU = \nabla_U Z$. The proof of this fact can be found in [3]. As a consequence of this result:

Proposition 3.3. *If X is a Killing non-vertical vector field on a Sasakian manifold (M, α, Z, J) and $[X, Z] = 0$, then $g(X, Z)$ cannot be constant.*

Proof. Suppose $g(X, Z)$ is constant. Then $X = g(X, Z)Z + B$ with B horizontal Killing and $\nabla_Z B = JB$. Therefore $[X, Z] = \nabla_X Z - \nabla_Z X = -JB - JB \neq 0$. □

3.3 Perturbation of Sasakian structures

Proposition 3.4. *Let (α, Z, J, g) be K-contact structure tensors on a manifold M. Let U be a Killing vector field such that $[U, Z] = 0$, $L_U\alpha = 0$ and $\alpha(U) > 0$. Then the vector field U is the characteristic vector field of a K-contact form β on M. Moreover, if α is a sasakian form, then so is β.*

Proof. Define new structure tensors : $\beta = \frac{\alpha}{\alpha(U)}$; for any vector fields X and Y on M, $\phi X = J(X - \beta(X)U)$ and $b(X, Y) = \frac{1}{\alpha(U)} g(X - \beta(X)U, Y - \beta(Y)U) + \beta(X)\beta(Y)$. We will verify that (β, U, ϕ, b) are K-contact structure tensors. β is obviously a contact form and $\beta(U) = 1 = b(U, U)$.

$$\begin{aligned}
\phi^2 X &= \phi[JX - \beta(X)JU] \\
&= J^2X - \beta(X)J^2U \\
&= -X + \alpha(X)Z - \beta(X)[-U + \alpha(U)Z] \\
&= -X + \beta(X)U + \alpha(X)Z - \frac{\alpha(X)}{\alpha(U)}\alpha(U)Z \\
&= -X + \beta(X)U.
\end{aligned}$$

Also

$$\begin{aligned}
i_U d\beta &= i_U d(\frac{\alpha}{\alpha(U)}) \\
&= i_U[-\frac{1}{\alpha(U)^2} d i_U\alpha \wedge \alpha + \frac{1}{\alpha(U)} d\alpha] \\
&= i_U[\frac{1}{\alpha(U)^2} i_U d\alpha \wedge \alpha + \frac{1}{\alpha(U)} d\alpha] \\
&= -\frac{\alpha(U)}{\alpha(U)^2} i_U d\alpha + \frac{1}{\alpha(U)} i_U d\alpha = 0.
\end{aligned}$$

This shows that U is the characteristic vector field of β. Next, we verify that b is a contact metric adapted to β and ϕ.

$$\begin{aligned}
b(X, \phi Y) &= \frac{1}{\alpha(U)} g(X - \beta(X)U, \phi Y) \\
&= \frac{1}{\alpha(U)} g(X, JY) - \beta(Y)JU) - \frac{\beta(X)}{\alpha(U)} g(X, JY - \beta(X)JU) \\
&= \frac{1}{\alpha(U)} g(X, JY) - \frac{\beta(Y)}{\alpha(U)} g(X, JU) - \frac{\beta(X)}{\alpha(U)} g(U, JY) \\
&= \frac{1}{\alpha(U)} \frac{1}{2} d\alpha(X, Y) - \frac{\alpha(Y)}{\alpha(U)^2} \frac{1}{2} d\alpha(X, U) - \frac{\alpha(X)}{\alpha(U)^2} \frac{1}{2} d\alpha(U, Y) \\
&= \frac{1}{\alpha(U)} \frac{1}{2} d\alpha(X, Y) + \frac{1}{2} \frac{1}{\alpha(U)^2} i_U d\alpha \wedge \alpha(X, Y) \\
&= \frac{1}{2}(\frac{1}{\alpha(U)} d\alpha + d(\frac{1}{\alpha(U)}) \wedge \alpha)(X, Y) \\
&= \frac{1}{2} d\beta(X, Y).
\end{aligned}$$

Finally, since $L_U\alpha = 0$ and $[U, Z] = 0$, we have automatically $L_U b = 0$ in view of the definition of b. Hence β is a K-contact form. Now, assuming that J is normal, that is, for any tangent vector fields X and Y, $[J, J](X, Y) + d\alpha(X, Y)Z = 0$, let X and Y satisfy $\beta(X) = 0 = \beta(Y)$ first. On those kind of vector fields, it is clear that ϕ and J coincide. Therefore,

$$\begin{aligned}
[\phi,\phi](X,Y) + d\beta(X,Y)U &= \phi^2([X,Y]) + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + d\beta(X,Y)U \\
&= -[X,Y] + [JX, JY] - \phi[JX, Y] - \phi[X, JY] \\
&= -[X,Y] + [JX, JY] - J[JX, Y] - J[X, JY] + (\beta([JX,Y] + [X, JY])JU \\
&= [J,J](X,Y) + d\alpha(X,Y)Z + \frac{1}{\alpha(U)}\alpha([JX,Y] + [X,JY])JU = 0
\end{aligned}$$

Next, we compute $[\phi, \phi](X, U)$, using the fact that U preserves J in the process.

$$\begin{aligned}
[\phi,\phi](X,U) &= -[X,U] + \beta([X,U])U - \phi[JX,U] \\
&= -[X,U] + J^2[U,X] = 0.
\end{aligned}$$

Since obviously, $[\phi, \phi](U, U) + d\beta(U, U) = 0$, we conclude from the above calculations that ϕ is also normal, hence β is a sasakian form. □

3.4 Basic cohomology

By $\mathbf{C}_b^p(Z)$ we denote the spaces of closed, basic p-forms on a contact manifold (M, α, Z). A p-form ω is said to be basic if $\omega(Z, X_1, ..., X_{p-1}) = 0$ for any $p-1$ vector fields $X_1, ..., X_{p-1}$ and $L_Z\omega = 0$. A p-form ω will be said to be basic exact if ω is basic and $\omega = d\mu$ where μ is a basic $p-1$-form. We denote by $\mathbf{B}_b^p(Z)$ the space of basic exact p-forms on M. The p-th basic cohomology group $H_b^p(Z)$ of (M, α, Z) is defined to be the quotient $H_b^p(Z) = \mathbf{C}_b^p(Z)/\mathbf{B}_b^p(Z)$.

Lemma 3.5. *Let μ be a harmonic 1-form on a K-contact manifold (M, α, Z, J, g). Then μ is a basic 1-form.*

Proof. Denote by ψ_t the 1-parameter group of isometries generated by Z. Since harmonic forms pull back into harmonic forms under isometries, we have that for all t, $\psi_t^*\mu$ is a harmonic 1-form which is co-homologous to μ, hence, by Hodge's Decomposition Theorem, $\psi_t^*\mu = \mu$ for all t. As a consequence $L_Z\mu = \frac{d}{dt}_{|t=0}\psi_t^*\mu = 0$. Since $L_Z\mu = i_Zd\mu + di_Z\mu = d(\mu(Z))$, it follows that $\mu(Z)$ is constant. We need to prove that $\mu(Z) = 0$. Suppose on the contrary that $\mu(Z) = k$ where k is a nonzero constant. Let $\beta = \frac{1}{k}\mu$. The 1-form β is a harmonic, nonsingular 1-form with $\beta(Z) = 1$. The 1-form $\gamma = \alpha - \beta$ satisfies $d\alpha = d\gamma$, hence a volume form for M is given by:

$$\alpha \wedge (d\alpha)^n = \alpha \wedge d\gamma \wedge (d\alpha)^{n-1} = -d(\alpha \wedge \gamma \wedge (d\alpha)^{n-1} + d\alpha \wedge \gamma \wedge (d\alpha)^{n-1}.$$

The form $d\alpha \wedge \gamma \wedge (d\alpha)^{n-1}$ is a basic, $2n+1$-form,hence is is identically zero. We have reached the contradiction that the volume form $\alpha \wedge (d\alpha)^n$ is exact on a closed manifold M and the proof of the lemma is complete. □

Proposition 3.6. *The first basic co-homology group $H_b^1(Z)$ of a closed K-contact manifold (M, α, Z, J, g) is isomorphic with the first DeRham co-homology group $H^1(M)$.*

Proof. The natural map $H_b^1(Z) \to H^1(M)$ is injective. Indeed, any basic 1-form $\eta = df$ represents a zero basic co-homology class, that is, η is basic exact due to the fact that $df(Z) = 0$ if and only if f is constant along Z. By a previous lemma, any harmonic 1-form μ on M is basic. This provides an injective linear map $H^1(M) \to H_b^1(Z)$ which must be an isomorphism. □

On compact Sasakian M^{2n+1}, the Betti numbers B_p are known to be even for odd p, $1 \leq p \leq n$ [7]. As a consequence, $S^1 \times S^{2n}$ and odd-dimensional tori carry no Sasakian structures. As a consequence of Proposition 3.6, we can extend this statement to K-contact manifolds as follows.

Corollary 3.7. *No torus T^{2n+1} carries a K-contact form.*

In [2], Blair and Goldberg showed that on a compact Sasakian manifold M^{2n+1}, there are no nonzero parallel p-forms for $1 \leq p \leq 2n$. This result extends to K-contact manifolds. First, as a consequence of Proposition 3.6, one has the

Proposition 3.8. *On a closed K-contact manifold, there can be no nonzero parallel 1-form.*

Proof. Let U be a parallel vector field. Then U is harmonic, $[U, Z] = 0$ and U is horizontal Killing, which is a contradiction to Proposition 3.3. □

Next, it is also easily extended to 2-forms as follows.

Proposition 3.9. *There cannot be any non-trivial parallel 2-form on a closed K-contact manifold.*

Proof. First observe that $L_Z\mu = 0$ for any harmonic (2-) form. Next, from

$$0 = L_z\mu(A, Z) = Z\mu(A, Z) - \mu([Z, A], Z) = \mu(\nabla_A Z, Z) = -\mu(JA, Z)$$

we deduce that $i_Z\mu = 0$; that is μ is basic. Next, for any A ,B,

$$0 = B\mu(Z, A) = \mu(\nabla_B Z, A) + \mu(Z, \nabla_B A) = -\mu(JB, A).$$

We see that μ must be identically zero. □

This result follows also from the work of Sharma [6]. More generally, on K-contact manifolds, closed or not, parallel forms can only be found in degrees 0 and 2n+1, as stated in the following theorem which was proved in [5].

Theorem 3.10. *On a K-contact manifold M^{2n+1} with K-contact form η and Reeb field Z, there are no nonzero parallel p-forms for $1 \leq p \leq 2n$.*

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