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Quelques résultats de contrôlabilité des systèmes linéaires sur les groupes de Heisenberg

Some controlability results for Linear Systems on Heisenberg Groups

Mouhamadou Dath* and Philippe Jouan†

Résumé

Le propos de cet article est d'énoncer des critères de contrôlabilité pour les systèmes linéaires sur les groupes Heisenberg généralisés. Plusieurs conditions suffisantes de contrôlabilité sont établies, et une étude de l'obstruction à la contrôlabilité est menée. Nous introduisons la notion de systèmes découplés dans le groupe Heisenberg de dimension cinq et nous obtenons des conditions nécessaires et suffisantes de contrôlabilité pour ces systèmes.

Mots clés : Groupes de Heisenberg généralisés ; Systèmes linéaires ; Contrôlabilité.

Abstract

This paper is intended to state controllability results for linear systems on generalized Heisenberg groups. Several sufficient controllability conditions are provided, and obstruction to controllability is studied. We introduce the notion of decoupled systems on the five dimensional Heisenberg group. Necessary and sufficient controllability conditions are obtained for these systems.

Key words: Generalized Heisenberg Group; Linear systems; Controllability.

1 Introduction

In this paper we are interested in controllability properties of linear systems on generalized Heisenberg groups, which are controlled systems

$$(\Sigma) \qquad \dot{g} = \mathcal{X}(g) + \sum_{j=1}^{m} u_j B_j(g)$$

where $\mathcal{X}$ is a linear vector field, that is a vector field whose flow is a one-parameter group of automorphisms, and the B_j's are right-invariant. Controllability means that for any pair $\{g, g'\}$ of points there exist a time T and a control function $t \longmapsto u(t)$ on $[0,T]$ such that the solution of (Σ) for this control function and for the initial point g reaches g' at time T. It is one of the main issues of geometric control theory because many other topics of this area only make sense for controllable systems.

The present paper follows and generalizes [2] where necessary and sufficient controllability conditions on the 2-dimensional affine group and the 3-dimensional Heisenberg group were stated. On generalized Heisenberg group we do not obtain such general necessary and sufficient conditions, but a series of sufficient ones and some obstruction criteria. However we were able to exhibit necessary and sufficient controllabilty conditions for decoupled systems in $\mathbb{H}^2$ (see Section 6).

*Département de Mathématiques et informatique, Faculté des Sciences et Techniques, Université Cheikh Anta Diop, Dakar, Sénégal. rassouldath@yahoo.fr.

†Lab. R. Salem, CNRS UMR 6085, Université de Rouen, avenue de l'université BP 12, 76801 Saint-Étienne-du-Rouvray France. philippe.jouan@univ-rouen.fr.

2 Basic definitions. Properties of linear systems

More details about linear vector fields and linear systems can be found in [4] and [5].

Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra (the set of right-invariant vector fields, identified with the tangent space at the identity). A vector field on G is said to be *linear* if its flow is a one-parameter group of automorphisms. Linear vector fields can also be characterized as follows : *A vector field $\mathcal{X}$ on a connected Lie group G is linear if and only if it belongs to the normalizer of $\mathfrak{g}$ in the algebra of analytic vector fields of G (that is $\forall Y \in \mathfrak{g}, \quad [\mathcal{X}, Y] \in \mathfrak{g}$) and verifies $\mathcal{X}(e) = 0$.* On account of this characterization, one can associate to a linear vector field $\mathcal{X}$ the derivation $D = -ad(\mathcal{X})$ of the Lie algebra $\mathfrak{g}$ of G. The minus sign in this definition comes from the formula $[Ax, b] = -Ab$ in $\mathbb{R}^n$. It also enables to avoid a minus sign in the useful formula :

$$\forall Y \in \mathfrak{g}, \quad \forall t \in \mathbb{R} \quad \varphi_t(\exp Y) = \exp(e^{tD}Y). \tag{1}$$

Throughout the paper the flow of a linear vector field $\mathcal{X}$ will be denoted by $(\varphi_t)_{t\in\mathbb{R}}$.

Definition 1. *A **linear system** on a connected Lie group G is a controlled system*

$$(\Sigma) \qquad \dot{g} = \mathcal{X}(g) + \sum_{j=1}^{m} u_j B_j(g)$$

where $\mathcal{X}$ is a linear vector field and the B_j's are right-invariant ones. The control $u = (u_1, \dots, u_m)$ takes its values in $\mathbb{R}^m$.

An input u being given (measurable and locally bounded), the corresponding trajectory of (Σ) starting from the identity e will be denoted by $e_u(t)$, and the one starting from the point g by $g_u(t)$. A straightforward computation shows that $g_u(t) = e_u(t)\varphi_t(g)$.

We denote by $\mathcal{A}(g, t) = \{g_u(t);\ u \in L^\infty[0, t]\}$ (resp. $\mathcal{A}(g, \leq t)$) (resp. $\mathcal{A}(g)$) the reachable set from g in time t (resp. in time less than or equal to t) (resp. in any time). In particular the reachable sets from the identity e are denoted by $\mathcal{A}_t = \mathcal{A}(e, t) = \mathcal{A}(e, \leq t)$ and $\mathcal{A} = \mathcal{A}(e)$. We also note $\mathcal{A}^- = \{g \in G;\ e \in \mathcal{A}(g)\}$ the set of points from which the identity can be reached. It is equal to the attainability set from the identity for the time-reversed system. Notice that (Σ) is controllable if and only if $\mathcal{A} = \mathcal{A}^- = G$. Now we analyze the **rank condition**, which is a very well-known necessary condition for controllability. Consider the Lie algebra generated by all the vector fields of the system. We recall that the rank condition means that the rank of that Lie algebra is equal to the dimension of the state space at all points.

Let V stand for the subspace of $\mathfrak{g}$ generated by $\{B_1, \dots, B_m\}$, let us denote by DV the smallest D-invariant subspace of $\mathfrak{g}$ that contains V, i.e. $DV = \text{Span}\{D^k Y;\ Y \in V \text{ and } k \in \mathbb{N}\}$, and let $\mathcal{LA}(DV)$ be the $\mathfrak{g}$ subalgebra generated by DV (as previously $D = -\text{ad}(\mathcal{X})$).

Proposition 1. *The subalgebra $\mathcal{LA}(DV)$ of $\mathfrak{g}$ is D-invariant. The system Lie algebra $\mathcal{L}$ is therefore equal to $\mathbb{R}\mathcal{X} \oplus \mathcal{LA}(DV)$, and the rank condition is satisfied if and only if $\mathcal{LA}(DV) = \mathfrak{g}$.*

Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}$ generated by $\{B_1, \dots, B_m\}$. It is a well known fact that we can replace the system by the extended one

$$(\widetilde{\Sigma}) \qquad \dot{g} = \mathcal{X}(g) + \sum_{j=1}^{p} u_j \widetilde{B}_j(g),$$

where $\widetilde{B}_1, \dots, \widetilde{B}_p$ is a basis of $\mathfrak{h}$, without modifying the closures of the sets $\overline{\mathcal{A}(g, \leq t)}$. It also well known that a system is locally controllable at an equilibrium point as soon as the linearized system is controllable (see [7] for instance). In this assertion "locally controllable" at a point g means that the set $\mathcal{A}(g, \leq t)$ is a neighbourhood of g for all $t > 0$. This leads to the following definition.

Definition and Proposition 2. *System (Σ) is said to satisfy the **ad-rank condition** if $D\mathfrak{h} = \mathfrak{g}$, in other words if the linearized system of $(\widetilde{\Sigma})$ is controllable. In that case the reachable set $\mathcal{A}_t$ is a neighbourhood of e for all $t > 0$.*

2.1 Controllability and quotient groups

The following propositions 3 and 4 concern closed subgroups and quotients. They are proved in [2] and of constant use in our approach.

Proposition 3. *Let H be a closed subgroup of G, globally invariant under the flow of $\mathcal{X}$. The system (Σ) on G, assumed to satisfy the rank condition, is controllable if and only if both conditions hold :*

1. *the system induced on G/H is controllable ;*
2. *the subgroup H is included in the closures $\overline{\mathcal{A}}$ and $\overline{\mathcal{A}^-}$ of $\mathcal{A}$ and $\mathcal{A}^-$.*

Proposition 4. *Let us assume that H is a connected and closed subgroup of G and that the restriction of $\mathcal{X}$ to H vanishes. Then H is included in $\mathcal{A}$ (resp. in $\mathcal{A}^-$) if and only if $\mathcal{A} \cap H$(resp. $\mathcal{A}^- \cap H$) is a neighbourhood of e in H.*

Singular and regular systems. The linear systems for which $\mathcal{X}$ vanishes on a connected, closed and $\mathcal{X}$-invariant (non trivial) subgroup will be referred to as *singular systems*, the other ones being *regular*. The previous proposition 4 is crucial in the singular case.

3 Controllability in $\mathbb{H}^1$

The reader is referred to the next section for the definition of $\mathbb{H}^1$.

We summarize here the main results of [2]. A one input system is equivalent by automorphism to a system in "normal form" in the canonical basis, that is : (Σ) $\quad \dot{g} = \mathcal{X}(g) + uX(g)$ where $D = \begin{pmatrix} 0 & b & 0 \\ 1 & d & 0 \\ 0 & f & d \end{pmatrix}$ is the matrix in the basis (X, Y, Z) of the derivation associated to $\mathcal{X}$. Notice that the controlled vector field is the first element of that basis. The main result of [2], is the following.

Theorem 1. *A system in normal form is controllable if and only if one of the following conditions hold :*

(i) $b < -\dfrac{d^2}{4}$,

(ii) $d = 0$ *and* $f \neq 0$,

Let us denote by (L) the classical linear system

$$(L) = \quad \begin{cases} \dot{x} = & by + u \\ \dot{y} = & x + dy \end{cases}$$

induced on the quotient $G/\mathbb{Z}(G)$. The eigenvalues of the matrix $\begin{pmatrix} 0 & b \\ 1 & d \end{pmatrix}$ are real if and only if $b \geq -\dfrac{d^2}{4}$. On the other hand (Σ) is singular if and only if $d = 0$ and the ad-rank condition is satisfied if and only if $f \neq 0$. Consequently Theorem 1 can be restated as :

Theorem 2. *The one-input system (Σ) on the Heisenberg group is controllable if and only if it satisfies the rank condition and*

(i) in the regular case : the eigenvalues of (L) are not real ;

(ii) in the singular case : the eigenvalues of (L) are not real or the ad-rank condition is satisfied.

4 Linear Systems on $\mathbb{H}^n$

The generalized Heisenberg group $\mathbb{H}^n$ is the $(2n+1)$-dimensional matrix subgroup of $GL(n+2, \mathbb{R})$ whose elements have the form :

$$\begin{pmatrix} 1 & y_1 & y_2 & \dots & y_n & z \\ 0 & 1 & 0 & \dots & 0 & x_1 \\ 0 & 0 & 1 & \dots. & 0 & x_2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \dots & 1 & x_n \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \qquad x_i, y_i, z \in \mathbb{R}.$$

Its Lie algebra $\mathfrak{h}^n$ is generated by the $2n+1$ right-invariant vector fields $X_1,\dots,X_n$, $Y_1,\dots,Y_n$, and Z defined by

$$\begin{cases} X_i &= E_{i+1,n+2} \qquad 1 \le i \le n \\ Y_i &= E_{1,i+1} + x_i E_{1,n+2} \qquad 1 \le i \le n \\ Z &= E_{1,n+2} \end{cases}$$

where E_{ij} is the matrix whose all entries vanish, excepted the rank i and column j one which is equal to 1. In canonical coordinates these vector fields write :

$$X_i = \frac{\partial}{\partial x_i}, \quad Y_i = \frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z} \tag{2}$$

4.1 Symplectic basis

The basis of $\mathfrak{h}^n$ defined above satisfies the following Lie bracket relations :

$$[X_i, Y_i] = Y_i X_i - X_i Y_i = Z \quad \text{for} \quad i = 1, \dots, n \tag{3}$$

and all the other brackets vanish. In particular the center of $\mathfrak{h}^n$ is generated by the field Z, and is equal to the derived algebra $\mathcal{D}^1\mathfrak{h}^n$. In what follows any basis $(X_1, Y_1, \dots, X_n, Y_n, Z)$ of $\mathfrak{h}^n$ that satisfy the relations (3) will be referred to as a *symplectic basis*. The importance of such basis comes from the following proposition.

Proposition 5.

1. *If $X \in \mathfrak{h}^n \setminus D^1\mathfrak{h}^n$ then there exists a symplectic basis such that $X = X_1$.*
2. *If $X, Y \in \mathfrak{h}^n$ and $[X,Y] \neq 0$ then there exists a symplectic basis such that $X = X_1$ and $Y = Y_1$.*
3. *If $B_1, \dots, B_m$ are linearly independant in $\mathfrak{h}^n/\mathcal{D}^1\mathfrak{h}^n$ and $[B_i, B_j] = 0$ for all $i,j = 1, \cdots, m$, then there exists a symplectic basis such that $B_i = X_i$ for $i = 1, \cdots, m$.*

To any symplectic basis we can associate a coordinate system in which the vector fields of the basis write like in (2). We make a constant use of these coordinates in the sequel.

4.2 Derivations and linear fields on $\mathfrak{h}^n$

In this section we compute the derivations and linear vector fields oh $\mathbb{H}^n$.

Proposition 6. *An endomorphism D of $\mathfrak{h}^n$ is a derivation if and only if its matrix in any symplectic basis has the following form :*

$$\begin{pmatrix} A_{11} & -\widetilde{A}_{21}^T & \dots & -\widetilde{A}_{n1}^T & 0 \\ A_{21} & A_{22} & \dots & -\widetilde{A}_{n2}^T & 0 \\ . & . & . & . & . \\ A_{n1} & A_{n2} & \dots & A_{nn} & 0 \\ a_{2n+1,1} & a_{2n+1,2} & \dots & a_{2n+1,2n} & d \end{pmatrix}$$

where the A_{ij}'s are 2×2 matrices, $\widetilde{A}_{ij}^T$ stands for the transpose of the comatrix of A_{ij}, and $tr(A_{ii}) = d$ for $1 \le i \le n$.

In order to compute the linear vector field associated to a derivation D the elements of $\mathfrak{h}^n$ will be denoted by $A = \begin{pmatrix} 0 & y_A & z_A \\ 0 & 0 & x_A \\ 0 & 0 & 0 \end{pmatrix}$ where $x_A = (x_1, \dots, x_n)$ and $y_A = (y_1, \dots, y_n)$ belong to $\mathbb{R}^n$.

The canonical scalar product of $\mathbb{R}^n$ will be denoted by $\langle , \rangle$, so that if B is another element of $\mathfrak{h}^n$ then the matricial product AB writes merely : $AB = \langle y_A, x_B \rangle Z$. Notice also that the elements of $\mathbb{H}^n$ have the form $g = I + G$, where I is the identity matrix of size $n+2$ and G belongs to $\mathfrak{h}^n$ (but $I + G$ is not equal to $\exp(G)$).

Proposition 7. *Let D be a derivation of $\mathfrak{h}^n$. It is associated to a unique linear vector field $\mathcal{X}$ of $\mathbb{H}^n$, and $\mathcal{X}$ is equal, at the point $g = I + G$ of $\mathbb{H}^n$ to :*

$$\mathcal{X}(g) = DG - \frac{1}{2}dG^2 + \frac{1}{2}(G(DG) + (DG)G) = DG + \frac{1}{2}(\langle y, x\rangle + \langle x, y\rangle - d\langle x, y\rangle)Z.$$

where d is defined by $DZ = dZ$.

Proposition 7 shows that the vector field $\mathcal{X}$ is composed of a linear part DG and a quadratic one in the $2n$ variables $x_1, y_1, \dots, x_n, y_n$, but that only the last coordinate z depends on the quadratic part.

5 Sufficient controllability conditions

We are now in a position to state sufficient controllability conditions. We begin by relating the rank condition on a quotient group to the rank condition of (Σ).

Proposition 8. *Let $\mathfrak{g}$ be an ideal of $\mathfrak{h}^n$ invariant by D, and let G be the subgroup generated by $\mathfrak{g}$. Then G is a closed Lie subgroup of $\mathbb{H}^n$ and the quotient $\mathbb{H}^n/G$ is an Abelian simply connected Lie group. The induced system on $\mathbb{H}^n/G$ satisfies the rank condition as soon as (Σ) does. It is in that case controllable in exact time T for any $T > 0$.*

Thanks to that proposition we can state

Theorem 3. *The linear system is assumed to satisfy the rank condition.*

1. *If (Σ) satisfies the ad-rank condition and is singular (that is d, defined by $DZ = dz$, vanishes) then it is controllable.*
2. *If the invariant vector field Z belongs to the Lie algebra generated by $B_1, \dots, B_m$, then the system is controllable in exact time T for all $T > 0$.*
3. *If the controlled vectors $B_1, \dots, B_m$ are linearly independant and if $m \geq n+1$, then (Σ) is controllable in exact time T for all $T > 0$.*

Sketch. All the items are proved by considering a suitable subgroup of $\mathbb{H}^n$ and by applying Proposition 3 (and Proposition 4 for the first item). □

6 Decoupling in $\mathbb{H}^2$

In this section we consider systems with two inputs in $\mathbb{H}^2$, hence in dimension 5.

Definition 2. *A 2 input linear system in $\mathbb{H}^2$ is said to be decoupled if*

(i) the vectors B_1 and B_2 are linearly independant and $[B_1, B_2] = 0$;

(ii) there exists a symplectic basis (X_1,Y_1,X_2,Y_2,Z) of $\mathfrak{h}^2$ such that $X_i = B_i$ and the ideal span$\{B_i, Y_i, Z\}$ is invariant by D, for $i = 1, 2$.

These systems can be put in a normal form that allows to decide their controllability.

Lemma 1. *Let (Σ) be a two inputs decoupled system in $\mathbb{H}^2$. If it satisfies the rank condition, there exists a symplectic basis $\{B_1, Y_1, B_2, Y_2, Z\}$ such that the matrix of the derivation D be :*

$$D = \begin{pmatrix} 0 & b & 0 & 0 & 0 \\ 1 & d & 0 & 0 & 0 \\ 0 & 0 & 0 & b' & 0 \\ 0 & 0 & c' & d & 0 \\ 0 & f & 0 & f' & d \end{pmatrix} \qquad \text{with} \quad c' \neq 0.$$

Let G_j be the subgroup of $\mathbb{H}^2$ generated by $\{B_j, Y_j, Z\}$ (it is a closed Lie subgroup of $\mathbb{H}^2$, see [1]). Thanks to the particular form of D the system induces a system (Σ_j) on G_j for $j = 1, 2$. It is clear that G_j is nothing but the Heisenberg group $\mathbb{H}^1$ and (Σ_j) a linear system for which controllability criteria are known (see Section 3).

Theorem 4. *Let (Σ) be a two inputs decoupled system in $\mathbb{H}^2$ in the form of Lemma 1. It is assumed to satisfy the rank condition. Then it is controllable if and only if one the following conditions holds :*

(i) one of the systems (Σ_1) and (Σ_2) is controllable ;

(ii) none of the systems (Σ_1) and (Σ_2) is controllable but $c' < 0$.

Recall that under the rank condition a regular system on $\mathbb{H}^1$ is controllable if and only if the eigenvalues of the system induced on $\mathbb{H}^1/\mathbb{Z}(\mathbb{H}^1)$ are not real (see Theorem 1 and Corollary 2 in Section 3).

This condition is no longer necessary in $\mathbb{H}^2$. Indeed consider a regular system with two decoupled cells. If their eigenvalues are real these cells are non controllable. However if the coefficient c' negative, then (Σ) is controllable despite the existence of real eigenvalues in the quotient.

7 Obstruction to Controllability

It has been proved in Section 5 that the system is controllable as soon as the generator Z of $\mathcal{D}^1\mathfrak{h}^n$ belongs to Span$\{B_1, \dots, B_m\}$. The purpose being herein to state conditions of non controllability we assume that $Z \neq$ Span$\{B_1, \dots, B_m\}$. Thanks to that assumption, we can choose a symplectic basis $(X_1, Y_1, \dots, X_n, Y_n, Z)$ such that the B_j's belong to the subspace of $\mathfrak{h}^n$ generated by the X_i's and the Y_i's. In the associated coordinates, the differential equation satisfied by the last coordinate z does not depend on the controls, it is $\dot{z} = dz + l(x_1, y_1, \dots, x_n, y_n) + Q(x_1, y_1, \dots, x_n, y_n)$, where $dz + l(x_1, y_1, \dots, x_n, y_n)$ is the linear form that comes from the last line of D and Q is a quadratic form in the $2n$ variables $x_1, y_1, \dots, x_n, y_n$.

Theorem 5. *It is assumed that $Z \notin \mathcal{LA}\{B_1, \dots, B_m\}$ and $d \neq 0$. If the quadratic form Q is non negative, and if* $\ker(Q) \subset \ker(l)$*, then the system is not controllable.*

We can actually go further by considering some particular modification of the variable z. *A change of variable $z \mapsto w$ will be said to be admissible if it has the form $w = z + P(x_1, y_1, \dots, x_n, y_n)$, where P is a polynomial of degree 2 with no constant term.* It is clear that the differential equation satisfied by w has again the form $\dot{w} = dw + l'(x_1, y_1, \dots, x_n, y_n) + Q'(x_1, y_1, \dots, x_n, y_n)$, where l' is a linear form and Q' is a quadratic form in the $2n$ variables $x_1, y_1, \dots, x_n, y_n$.

Theorem 6. *It is assumed that $Z \notin \mathcal{LA}\{B_1, \dots, B_m\}$ and $d \neq 0$.*

If for some admissible change of variable the quadratic form Q' is non negative and $\ker(Q') \subset \ker(l')$, *then the system is not controllable.*

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