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Energy decay of a thermoelastic system with boundary condition of memory type

Stabilisation d'un système de la thermoélasticité avec conditions aux limites de type mémoire

EL Hadji Malick BA * and Abdoulaye Sène †

Abstract

Using the multiplier method and a suitable Lyapounov functional, we establish exponential decay of energy for an isotropic thermoelastic system subject to Dirichlet boundary condition on one part of the boundary and a memory dissipative condition at the other part of the boundary.

Key words: thermoelasticity, feedbacks, energy, stabilization.

Résumé

Nous établissons, par la méthode des multiplicateurs et une fonction de Lyapounov, la décroissance exponentielle d'un système d'un système thermoélastique avec feedback de type mémoire sur une partie de la frontière.

Mots clés : Thermoélésticité, feedbacks, énergie, stabilisation.

AMS Subject classification: 93D15, 93D05, 93C60.

1 Introduction

Let Ω be a non empty bounded open subset of $\mathbb{R}^n, n \geq 1$, with a boundary Γ of class C^2. We denote by $\nu = (\nu_1, \dots, \nu_n)$ the unit outward normal vector along Γ. For a fixed $x_0 \in \mathbb{R}^n$ we define the function $m(x) = x - x_0$; $x \in \mathbb{R}^n$ and the following partition of the boundary Γ:

$$\Gamma_1 = \{x \in \Gamma : m(x) \cdot \nu(x) \leq 0\}, \quad \Gamma_2 = \{x \in \Gamma : m(x) \cdot \nu(x) > 0\}. \tag{1}$$

In this paper we consider the system of isotropic thermo-elasticity:

$$\begin{cases} u'' - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u + \alpha\nabla\theta = 0 \text{ in } Q := \Omega \times \mathbb{R}^+, \\ \theta' - \Delta\theta + \beta \mathrm{div} u' = 0 \text{ in } Q, \\ \theta = 0 \text{ on } \Gamma \times \mathbb{R}^+, \quad u = 0 \text{ on } \Gamma_1 \times \mathbb{R}^+, \\ \mu\frac{\partial u}{\partial \nu} + (\lambda+\mu)\mathrm{div}(u)\nu + am\cdot\nu u + \int_0^t k(t-s)u'(.,s)ds + m\cdot\nu u' = 0 \text{ on } \Gamma_2 \times \mathbb{R}^+, \\ u(\cdot,0) = u_0, \quad u'(\cdot,0) = u_1, \quad \theta(\cdot,0) = \theta_0 \text{ in } \Omega, \end{cases} \tag{2}$$

where $u = u(x,t) = (u_1(x,t), \dots, u_n(x,t))$ denotes the displacement vector field while $\theta = \theta(x,t)$ is the temperature. The function a is non negative and belongs to $C^1(\Gamma_2)$; $k : [0; +\infty[\to \mathbb{R}$ is a function of

*Département de Mathématiques et Informatique, Faculté des Sciences et Techniques, Université Cheikh Anta Diop, Dakar, Sénégal. elmaodoba@yahoo.fr.

†Département de Mathématiques et Informatique, Faculté des Sciences et Techniques, Université Cheikh Anta Diop, Dakar, Sénégal. abdousen@ucad.sn.

class C^2. The coupling parameters α and β are supposed to be positive. Finally, λ and μ are Lame's constants.

The stabilization of different variant of the system (2) has been studied in the literature in the case $k \equiv 0$. Up to now, there are a lot of works on this topic, see, for example [7, 8, 12, 2, 4]. For boundary condition of memory type we can cite among others [11, 9, 10]. In [9, 10], the authors consider the boundary condition $u = 0$ on $\Gamma_1 \times \mathbb{R}^+$ and,

$$\int_0^t k(t-s)\Big[\mu\frac{\partial u(s)}{\partial \nu}u(s) + (\lambda+\mu)\mathrm{div}u(s)\nu\Big]ds + au'(t) + u(t) = 0 \text{ on } \Gamma_2 \times \mathbb{R}^+.$$

The aims of this work is to establish exponential decay for energy of system (2).

2 The main result

In the remainder of our paper we suppose that

$$\Gamma_1 \neq \emptyset \ \text{ or } \ 0 < a(x) < a_0, \ \forall x \in \Gamma_2, \tag{3}$$

where a_0 is a constant whose magnitude will be specified after. Furthermore, in order to avoid regularity problems related to the change of boundary conditions we assume that $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$. We finally suppose that there exist positive constants γ_0 and γ_1 such that

$$\left\{\begin{array}{lcl} k(t) & \geq & 0, \\ k'(t) & \leq & -\gamma_0 k(t), \\ k''(t) & \geq & -\gamma_1 k'(t). \end{array}\right. \tag{4}$$

The assumptions (4) are relatively standard (see [11, 1, 15]). We define the following Hilbert spaces:

$$H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega);\ u = 0 \ \text{ on } \Gamma_1\}, \tag{5}$$

$$W = (H^1_{\Gamma_1}(\Omega))^n \times (L^2(\Omega))^n, \tag{6}$$

$$\mathcal{H} = W \times L^2(\Omega). \tag{7}$$

The space W is equipped with the natural norm:

$$\|(u,v)\|_W^2 = \int_\Omega [|v|^2 + \mu|\nabla u|^2 + (\lambda+\mu)|\mathrm{div}u|^2]dx + \int_{\Gamma_2} am\cdot\nu|u|^2 d\Gamma.$$

With assumption (4), the system (2) can be formulated as an evolution integral equation of variational type [14]. Therefore the results from [11, 13] allow to state the following results.

Théorème 2.1. *Let Γ_1 and Γ_2 be given by (1) and satisfying (3) and $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$. Assume that the function k satisfy (4). Then for initial data (u_0, u_1, θ_0) in $\mathcal{H}$, the system (2) has a unique (weak) solution (u, θ) satisfying $(u, u', \theta) \in C([0,\infty); \mathcal{H})$.*

We define the energy of the system (2) by

$$\begin{aligned} E(t) &= \frac{1}{2}\int_\Omega \Big\{|u'|^2 + \mu|\nabla u|^2 + (\lambda+\mu)|\mathrm{div}u|^2 + \frac{\alpha}{\beta}|\theta|^2\Big\}dx \\ &+ \frac{1}{2}\int_{\Gamma_2} am\cdot\nu|u|^2 d\Gamma + \frac{1}{2}\int_{\Gamma_2} k(t)[u(t)-u(0)]^2 d\Gamma - \frac{1}{2}\int_{\Gamma_2}\int_0^t k'(t-s)[u(t)-u(s)]^2 ds d\Gamma. \end{aligned} \tag{8}$$

The main result of this paper is the next theorem.

Théorème 2.2. *Let Γ_1 and Γ_2 given by (1) and satisfying (3) and $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$. Assume that the functions k satisfy (4). Then there exist positive constants C_1 and C_2 such that the energy of any solution of (2) satisfies*

$$E(t) \leq C_1 e^{-C_2 t}, \quad \forall t > 0. \tag{9}$$

The rest of the paper is devoted to the proof of Theorem 2.2.

3 Proof of Theorem 2.2

Deriving (8) in time and integrating by parts in space, we readily see that

$$\begin{aligned} E'(t) &= -\frac{\alpha}{\beta}\int_\Omega |\nabla\theta(x,t)|^2\,dx - \int_{\Gamma_2} m\cdot\nu|u'|^2 d\Gamma + \frac{1}{2}\int_{\Gamma_2} k'(t)[u(t)-u(0)]^2 d\Gamma \\ &\quad -\frac{1}{2}\int_{\Gamma_2}\int_0^t k''(t-s)[u(t)-u(s)]^2 ds d\Gamma, \end{aligned} \tag{10}$$

and consequently,

$$\begin{aligned} E(T)-E(S) &= -\frac{\alpha}{\beta}\int_S^T\int_\Omega |\nabla\theta|^2 dxdt + \frac{1}{2}\int_{\Gamma_2}\int_S^T k'(t)[u(t)-u(0)]^2 dtd\Gamma \\ &\quad -\int_S^T\int_{\Gamma_2} m\cdot\nu|u'(t)|^2 dtd\Gamma - \frac{1}{2}\int_S^T\int_{\Gamma_2}\int_0^t k''(t-s)[u(t)-u(s)]^2 dsd\Gamma dt, \end{aligned}$$

$\forall\ 0\le S\le T<\infty$. The assumptions on k, k' and k'' lead to the decay of the energy.

Let us introduce the constant

$$R_0 = \max_{x\in\overline{\Omega}}\Big(\sum_{k=1}^n (x_k - x_{0k})^2\Big)^{1/2}. \tag{11}$$

Further let γ and λ_0 be the smallest positive constantes such that for all $u\in(H^1_{\Gamma_1}(\Omega))^n$

$$\int_{\Gamma_2}|u|^2 d\Gamma \le \gamma^2\left(\int_\Omega\{\mu|\nabla u|^2 + (\lambda+\mu)|\mathrm{div}u|^2\}dx + \int_{\Gamma_2} am\cdot\nu|u|^2 d\Gamma\right), \tag{12}$$

and

$$\|u\|^2_{(L^2(\Omega))^n} \le \lambda_0^2\left(\int_\Omega\{\mu|\nabla u|^2 + (\lambda+\mu)|\mathrm{div}u|^2\}dx + \int_{\Gamma_2} am\cdot\nu|u|^2 d\Gamma\right) \tag{13}$$

respectively. Consider the standard energy (corresponding to $k\equiv 0$)

$$\mathcal{E}(t) = \frac{1}{2}\int_\Omega\Big\{|u'|^2 + \mu|\nabla u|^2 + (\lambda+\mu)|\mathrm{div}u|^2 + \frac{\alpha}{\beta}|\theta|^2\Big\}dx + \frac{1}{2}\int_{\Gamma_2} am\cdot\nu|u|^2 d\Gamma. \tag{14}$$

To prove Theorem 2.2, we start by two technical lemmas.

Lemme 3.1. *Let (u,θ) be a strong solution of (2). Define $M(u) = 2(m\cdot\nabla)u + (n-1)u$. Then, for all $t\ge 0$, there exists a positive constant η such that $\|M(u)\|^2_{(L^2(\Omega))^n}\le\eta\mathcal{E}(t)$.*

Proof of Lemma 3.1. By integration by parts, we have

$$\begin{aligned} \|M(u)\|^2_{(L^2(\Omega))^n} &= \int_\Omega[|2(m\cdot\nabla)u|^2 + (n-1)^2|u|^2 + 4(n-1)u\cdot(m\cdot\nabla)u]dx \\ &= \int_\Omega[|2(m\cdot\nabla)u|^2 + (n-1)^2|u|^2 + 2(n-1)m\cdot\nabla(|u|^2)]dx \\ &= \int_\Omega[|2(m\cdot\nabla)u|^2 + (1-n^2)|u|^2]dx + 2(n-1)\int_{\Gamma_2} m\cdot\nu|u|^2 d\Gamma \\ &\le 4R_0^2\int_\Omega|\nabla u|^2 dx + 2(n-1)\int_{\Gamma_2} m\cdot\nu|u|^2 d\Gamma. \end{aligned}$$

We conclude using Korn's inequality. □

Let us define, for all $t>0$, the function $\varphi_\sigma(t) = \int_\Omega\left[u'M(u) + \frac{\sigma}{2}\theta^2\right]dx$, where (u,θ) is strong solution of (2) and σ a positive constant.

Lemme 3.2. *For a suitable parameter σ, there exists two positive constants c_0 and C such that for all regular (u,θ) strong solution of (2)*

$$\begin{aligned} \varphi_\sigma'(t) &\leq C\left(\int_{\Gamma_2}|u'(t)|^2d\Gamma+\int_{\Gamma_2}k(t)[u(t)-u(0)]^2d\Gamma-\int_{\Gamma_2}\int_0^t k'(t-s)|u(t)-u(s)|^2dsd\Gamma\right)\\ &+\int_{\Gamma_2}am\cdot\nu u^2d\Gamma-c_0\mathcal{E}(t). \end{aligned} \tag{15}$$

Proof. Differentiating $\varphi_\sigma(t)$ we have $\varphi_\sigma'(t)=\int_\Omega[u''M(u)+u'(2m\cdot\nabla u'+(n-1)u')+\sigma\theta'\theta]dx$. Note that $M(u)=(M_i)_{1\leq i\leq n}$ where $M_i=2m_k\dfrac{\partial}{\partial x_k}u_i+(n-1)u_i$ (we use repetited convention indice). Appling Green's formula, we have

$$\begin{aligned} \int_\Omega \Delta u_iM_idxdt &= 2\int_\Gamma\frac{\partial u_i}{\partial\nu}m_k\frac{\partial u_i}{\partial x_k}d\Gamma-\int_\Gamma m_k\nu_k|\nabla u_i|^2d\Gamma\\ &+(n-2)\int_\Omega|\nabla u_i|^2dx+(n-1)\int_\Gamma\frac{\partial u_i}{\partial\nu}u_id\Gamma-(n-1)\int_\Omega|\nabla u_i|^2dx\\ &= 2\int_\Gamma\frac{\partial u_i}{\partial\nu}m_k\frac{\partial u_i}{\partial x_k}d\Gamma-\int_\Gamma m_k\nu_k|\nabla u_i|^2d\Gamma+(n-1)\int_\Gamma\frac{\partial u_i}{\partial\nu}u_id\Gamma-\int_\Omega|\nabla u_i|^2dx. \end{aligned} \tag{16}$$

$$\begin{aligned} \int_\Omega\frac{\partial}{\partial x_i}(\mathrm{div}u)M_idx &= 2\int_\Gamma \mathrm{div}u\, m_k\frac{\partial u_i}{\partial x_k}\nu_id\Gamma-\int_\Gamma m_k\nu_k|\mathrm{div}u|^2d\Gamma\\ &+(n-2)\int_\Omega|\mathrm{div}u|^2dx+(n-1)\int_\Gamma\mathrm{div}u\,u_i\nu_id\Gamma-(n-1)\int_\Omega|\mathrm{div}u|^2dx\\ &=2\int_\Gamma\mathrm{div}u\,m_k\frac{\partial u_i}{\partial x_k}\nu_id\Gamma-\int_\Gamma m_k\nu_k|\mathrm{div}u|^2d\Gamma+(n-1)\int_\Gamma\mathrm{div}u\,u_i\nu_id\Gamma-\int_\Omega|\mathrm{div}u|^2dx. \end{aligned} \tag{17}$$

Using these different identities, we obtain

$$\begin{aligned} \frac{d}{dt}\left(\int_\Omega u'M(u)\right) &= -\int_\Omega\left[|u'|^2+\mu|\nabla u|^2+(\lambda+\mu)|\mathrm{div}u|^2\right]dx+\int_\Gamma\left(\mu\frac{\partial u}{\partial\nu}+(\lambda+\mu)\mathrm{div}u\cdot\nu\right)M(u)d\Gamma\\ &-\int_\Omega\alpha\nabla\theta M(u)dx+\int_\Gamma m\cdot\nu|u'|^2d\Gamma-\int_\Gamma m\cdot\nu\left[\mu|\nabla u|^2+(\lambda+\mu)|\mathrm{div}u|^2\right]d\Gamma. \end{aligned} \tag{18}$$

In the same way, by using the second identity of the system (2), it appears that

$$\int_\Omega\theta'\theta dx=-\int_\Omega|\nabla\theta|^2dx-\beta\int_\Omega\theta\mathrm{div}u'dx. \tag{19}$$

Grouping these above inequalities and proceeding as in [11], we obtain

$$\begin{aligned} \varphi_\sigma'(t) &\leq -2\mathcal{E}(t)+\frac{\alpha}{\beta}\int_\Omega\theta^2dx+\int_{\Gamma_2}am\cdot\nu|u|^2d\Gamma+\int_\Gamma\left(\mu\frac{\partial u}{\partial\nu}+(\lambda+\mu)\mathrm{div}u\cdot\nu\right)M(u)d\Gamma\\ &-\int_\Gamma m\cdot\nu\left[\mu|\nabla u|^2+(\lambda+\mu)|\mathrm{div}u|^2\right]d\Gamma+\int_\Gamma m\cdot\nu|u'|^2d\Gamma\\ &-\int_\Omega\alpha\nabla\theta M(u)dx-\sigma\int_\Omega|\nabla\theta|^2dx+\sigma\beta\int_\Omega\nabla\theta u'dx. \end{aligned} \tag{20}$$

Taking into account the boundary conditions (2) (implying in particular $\dfrac{\partial u_i}{\partial x_k}=\dfrac{\partial u_i}{\partial\nu}\nu_k$ on Γ_1), we arrive at

$$\begin{aligned} \varphi_\sigma'(t) &\leq -2\mathcal{E}(t)+\frac{\alpha}{\beta}\int_\Omega\theta^2dx+\int_{\Gamma_2}am\cdot\nu|u|^2d\Gamma+\int_{\Gamma_2}\left(\mu\frac{\partial u}{\partial\nu}+(\lambda+\mu)\mathrm{div}u\cdot\nu\right)M(u)d\Gamma\\ &+\int_{\Gamma_2}m\cdot\nu|u'|^2d\Gamma-\int_\Omega\alpha\nabla\theta M(u)dx-\sigma\int_\Omega|\nabla\theta|^2dx+\sigma\beta\int_\Omega u'\nabla\theta dx \end{aligned} \tag{21}$$

By Young's formula and Lemma 3.1, we have for all $\varepsilon > 0$ the estimation

$$\begin{aligned}\varphi_\sigma^{'}(t) \;\leq\; & (-2+\varepsilon\eta+\varepsilon\sigma\beta)\mathcal{E}(t)+\int_{\Gamma_2} am\cdot\nu|u|^2d\Gamma+\int_{\Gamma_2} m\cdot\nu|u'|^2d\Gamma+\frac{c_1}{\varepsilon}\int_{\Gamma_2}[\mu\frac{\partial u}{\partial\nu}+(\lambda+\mu)\mathrm{div}u\cdot\nu]^2d\Gamma \\ &+\varepsilon\int_{\Gamma_2}(|\nabla u|^2+|u|^2)d\Gamma+\frac{\alpha}{\beta}\int_\Omega\theta^2dx+\left(\frac{\alpha^2+1}{4\varepsilon}-\sigma\right)\int_\Omega|\nabla\theta|^2dx \qquad (22)\end{aligned}$$

where c_1 is a positive constant. Using Poincaré's Theorem and trace inequality, from (22), by choosing ε sufficiently small, we obtain

$$\varphi_\sigma'(t)\leq -c_2\mathcal{E}(t)+\int_{\Gamma_2} m\cdot\nu|u|^2d\Gamma+c_3\int_{\Gamma_2}\left[\mu\frac{\partial u}{\partial\nu}+(\lambda+\mu)\mathrm{div}u\cdot\nu\right]^2d\Gamma+\frac{\alpha}{\beta}\int_\Omega\theta^2dx+\left(\frac{\alpha^2+1}{4\varepsilon}-\sigma\right)\int_\Omega|\nabla\theta|^2dx.$$

for two suitable constants c_2 and c_3. Now, we take σ such that

$$\frac{\alpha}{\beta}\int_\Omega\theta^2dx+(\frac{\alpha^2+1}{4\varepsilon}-\sigma)\int_\Omega|\nabla\theta|^2dx<0.$$

We arrive at

$$\varphi_\sigma^{'}(t)\leq -c_2\mathcal{E}(t)+\int_{\Gamma_2} m\cdot\nu|u|^2d\Gamma++c_3\int_{\Gamma_2}[\mu\frac{\partial u}{\partial\nu}+(\lambda+\mu)\mathrm{div}u\cdot\nu]^2d\Gamma. \qquad (23)$$

Note that as in [11] the boundary condition on Γ_2 in system (2) may be rewritten as

$$\mu\frac{\partial u}{\partial\nu}+(\lambda+\mu)\mathrm{div}u\nu+am\cdot\nu u+m\cdot\nu u'-\int_0^t k'(t-s)[u(t)-u(s)]ds-k(t)[u(t)-u(0)]=0 \qquad (24)$$

and from Cauchy-Schwarz's inequality

$$\left(\int_0^t k'(t-s)[u(t)-u(s)]ds\right)^2\leq[k(0)-k(t)]\int_0^t(-k'(t-s))[u(t)-u(s)]^2ds. \qquad (25)$$

Then using (24) and (25) there exists a positive constant C

$$\begin{aligned}\int_{\Gamma_2}[\mu\frac{\partial u}{\partial\nu}+(\lambda+\mu)\mathrm{div}u\nu]^2d\Gamma \;\leq\; & C\left[\int_{\Gamma_2}|u'|^2d\Gamma-\int_{\Gamma_2}\int_0^t k'(t-s)[u(t)-u(s)]^2dsd\Gamma\right. \\ &\left.+\int_{\Gamma_2}k(t)[u(t)-u(0)]^2d\Gamma+\int_{\Gamma_2}u^2d\Gamma\right], \qquad (26)\end{aligned}$$

that substituted in (23) gives (15). □

Proof of Theorem 2.2. By the assumptions on k, k' and k'', we can obtain from the identity (10) the estimate

$$\begin{aligned}E'(t) \;\leq\; & -\frac{\alpha}{\beta}\int_\Omega|\nabla\theta(x,t)|^2dx-\int_{\Gamma_2}m\cdot\nu|u'|^2d\Gamma-\frac{\gamma_0}{2}\int_{\Gamma_2}k(t)[u(t)-u(0)]^2d\Gamma \\ &+\frac{\gamma_1}{2}\int_{\Gamma_2}\int_0^t k^{'}(t-s)[u(t)-u(s)]^2dsd\Gamma. \qquad (27)\end{aligned}$$

Define the Lyapounov functionnal $E^*(t):=E(t)+\rho\varphi_\sigma(t)$ where ρ is a positive constant chosen sufficiently small later on and σ be such that Lemma 3.2 hold. Then, by the identity (10) and Lemma 3.2 , for ρ sufficiently small we have

$$(E^*)^{'}(t)\leq -c_0\rho\mathcal{E}(t)+\rho\int_{\Gamma_2}am\cdot\nu u^2d\Gamma-\frac{\gamma_0}{4}\int_{\Gamma_2}k(t)[u(t)-u(0)]^2d\Gamma+\frac{1}{4}\int_{\Gamma_2}\int_0^t k^{'}(t-s)[u(t)-u(s)]^2dsd\Gamma \qquad (28)$$

from which follows, for a_0 (from (3)) sufficiently small,

$$(E^*)^{'}(t)\leq -c_4E(t), \quad \forall t>0 \qquad (29)$$

for a suitable constant c_4. Observing that $E^*(t)\leq c_5E(t)$ for a suitable positive constant c_5, inequality (29) implies $(E^*)^{'}(t)\leq -c_4E^*(t)$. This means that E^* is exponentially decreasing. Therefore, since $E(t)\leq cE^*(t)$ for a suitable constant c, the Theorem 2.2 immediately follows. □

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